April 24 MATH 1112 sec. 54 Spring 2019

Section 8.1: The Laws of Sines and Cosines

In order to use the Law of Sines, we must know one angle-side pair (e.g. A and a). Since each angle is greater than 0° and less than 180°, all sine values are positive. So the law can be stated as

Law of Sines:

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

or

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

The Law of Sines can be used for AAS, ASA, or SSA. We still have to consider the SSA case.

Law of Sines: Ambiguous Case (SSA)

Given two sides and an angle **not between them**, the information may correspond to one triangle, two triangles, or no triangles.

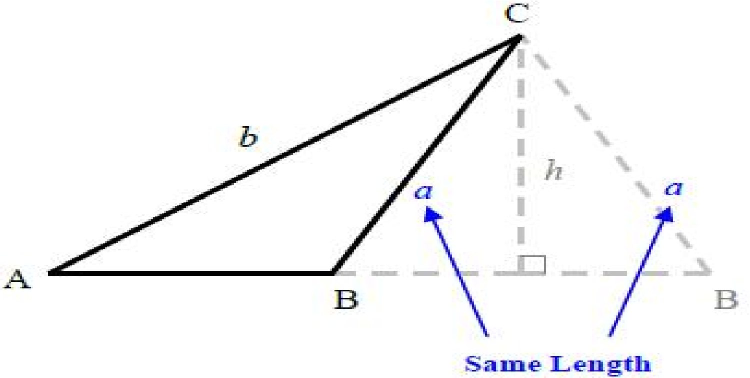


Figure: If $a > h = b \sin A$, there can be two different triangles with the same a, b, and A. Since B isn't known, it might be acute or obtuse. Since c isn't known, it may be *long* or *short*.

Law of Sines: Ambiguous Case (SSA)

$\sin(180° - \theta)$

$= \sin\theta$

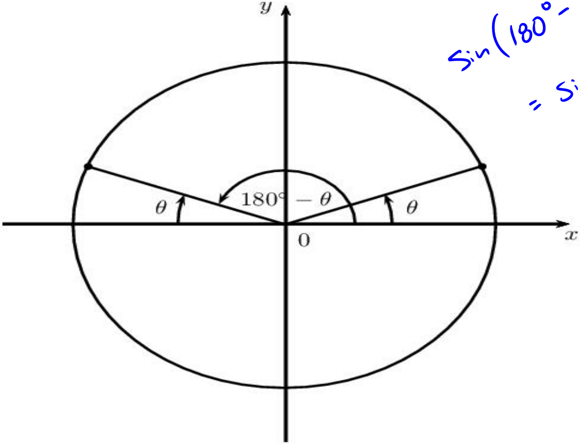


Figure: Remember that $\sin(180^\circ - \theta) = \sin\theta$. So if we know the number $\sin B$, it's not clear if we just have one angle B, or two angles B and $180^\circ - B$.

Law of Sines: Ambiguous Case (SSA)

We don't have to *memorize* cases! We can always check to see if zero, one, or two triangles exist.

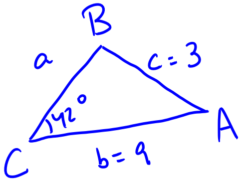
Suppose we know sides a and b and the angle A. Then

$$\text{set} \quad \sin B = \frac{b \sin A}{a}.$$

- If this number is > 1, there are **no triangles**. (A sine can't be bigger than 1!)
- Otherwise, let $B_1 = \sin^{-1}\left(\frac{b \sin A}{a}\right)$ and let $B_2 = 180^\circ - B_1$.
- If $B_2 + A \geq 180^\circ$, then there is exactly **one triangle**.
- If $B_2 + A < 180^\circ$ there are **two triangles**. (There's still room for a third angle.)

Exampe 1 (SSA)

Solve for all possible triangles. $C = 42^\circ, \quad b = 9, \quad c = 3$



we have the side-angle pair

C & c

By Law of Sines

$$\frac{\sin B}{b} = \frac{\sin C}{c} \Rightarrow \sin B = \frac{b \sin C}{c}$$

$$\sin B = \frac{9 \sin 42^\circ}{3} \approx 2.007$$

As $\sin B$ can't be greater than 1,

there is no angle B.

No triangle has $C=42^0$, $b=9$ and $c=3$.

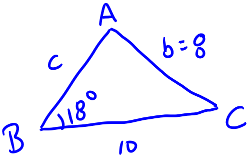
Recall $-1 \le \sin x \le 1$

ditto

$-1 \le \cos x \le 1$

Exampe 2 (SSA)

Solve for all possible triangles. $B = 18^\circ, \quad b = 8, \quad a = 10$



we know the B, b pair

$$\frac{\sin A}{a} = \frac{\sin B}{b} \Rightarrow \sin A = \frac{a \sin B}{b}$$

$$\sin A = \frac{10 \sin 18^\circ}{8} \approx 0.3863$$

an A is $A = \sin^{-1}(0.3863) \approx 22.7^\circ$

Another possible angle with this sine value is

$$180^\circ - 22.7^\circ \approx 157.3^\circ$$

Question: Could there be 2 triangles, one with $A = 22.7^\circ$ and another with $A = 157.3^\circ$?

There are 2 triangles if $B + A$ is less than 180° for both A values.

$$B + 157.3^\circ = 18^\circ + 157.3^\circ = 175.3^\circ < 0$$

There is room for a third angle, hence there are 2 triangles.

Triangle w/ $A = 22.7^\circ$

$$C = 180^\circ - A - B = 180^\circ - 22.7^\circ - 18^\circ = 139.3^\circ$$

$$\frac{c}{\sin C} = \frac{b}{\sin B} \Rightarrow c = \frac{b\sin C}{\sin B} = \frac{8\sin(139.3^\circ)}{\sin 18^\circ}$$

$$\approx 16.9$$

For $A = 157.3^\circ$, $C = 180^\circ - 157.3^\circ - 18^\circ = 4.7^\circ$

$$c = \frac{b\sin C}{\sin B} = \frac{8\sin(4.7^\circ)}{\sin(18^\circ)} \approx 2.1$$

Triangle 1: $A = 22.7^\circ$, $B = 18^\circ$, $C = 139.3^\circ$

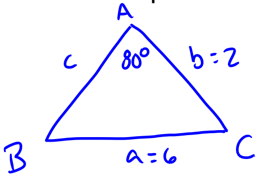
$a = 10$ $\quad b = 8$ $\quad c = 16.9$

triangle 2: $A = 157.3^\circ$ $\quad B = 18^\circ$ $\quad C = 4.7^\circ$

$a = 10$ $\quad b = 8$ $\quad c = 2.1$

Exampe 3 (SSA)

Solve for all possible triangles. $A = 80^\circ, \quad a = 6, \quad b = 2$



Our side angle pair is A-a

$$\frac{\sin B}{b} = \frac{\sin A}{a} \Rightarrow \sin B = \frac{b \sin A}{a}$$

$$\sin B = \frac{2\sin 80^\circ}{6} \approx 0.3283$$

$$B \approx \sin^{-1}(0.3283) \approx 19.2^\circ$$

Another possible B is $B = 180^\circ - 19.2^\circ = 160.8^\circ$

Are there 2 triangles?

$$80^\circ + 160.8^\circ = 240.8^\circ > 180^\circ$$

There is only one triangle.

For $A = 80^\circ$, $B = 19.2^\circ$,

$$C = 180^\circ - 80^\circ - 19.2^\circ = 80.8^\circ$$

And $\dfrac{c}{\sin C} = \dfrac{a}{\sin A} \Rightarrow c = \dfrac{a \sin C}{\sin A} = \dfrac{6 \sin(80.8^\circ)}{\sin 80^\circ}$

$$\approx 6.01$$

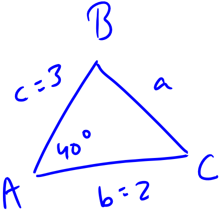
$A = 80^\circ$, $B = 19.2^\circ$, $C = 80.8^\circ$

$a = 6$ $\quad b = 2$ $\quad c = 6.01$

Section 8.2: The Law of Cosines

Suppose we wish to solve a triangle given

$$A = 40^\circ, \quad b = 2, \quad c = 3$$



$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

we don't have any side / angle pair.

The law of sines won't help.

The Law of Cosines

Theorem: For the triangle labeled using the previous convention, all three of the following equations hold

$$\begin{aligned} a^2 &= b^2 + c^2 - 2bc\cos A \\ b^2 &= a^2 + c^2 - 2ac\cos B \\ c^2 &= a^2 + b^2 - 2ab\cos C \end{aligned}$$

This can be used if two sides and an included angle (SAS) are known, or if the three sides (SSS) are known.

Example (SAS)

Solve the triangle given $A = 40^\circ$, $b = 2$, $c = 3$

we can use

$$a^2 = b^2 + c^2 - 2bc\cos A$$

$$= 2^2 + 3^2 - 2(2)(3)\cos 40^\circ \approx 3.807$$

So $a = \sqrt{3.807} \approx 1.95$

We can find B from the Law of Sines

$$\frac{\sin B}{b} = \frac{\sin A}{a} \Rightarrow \sin B = \frac{b\sin A}{a}$$

$$\sin B = \frac{2\sin 40^\circ}{1.95} \approx 0.6588$$

$$B \approx \sin^{-1}(0.6588) \approx 41.2^\circ$$

Then $C = 180^\circ - A - B = 180^\circ - 40^\circ - 41.2^\circ \approx 98.8^\circ$

$A = 40^\circ \quad B = 41.2^\circ \quad C = 98.8^\circ$

$a = 1.95 \quad b = 2 \quad c = 3$