April 24 Math 2306 sec. 53 Spring 2019

Section 18: Sine and Cosine Series

Functions with Symmetry

If f is even on $(-p, p)$, then the Fourier series of f has only constant and cosine terms. Moreover

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{p}\right)$$

where

$$a_0 = \frac{2}{p}\int_0^p f(x)\,dx$$

and

$$a_n = \frac{2}{p}\int_0^p f(x)\cos\left(\frac{n\pi x}{p}\right)\,dx.$$

Fourier Series of an Odd Function

If f is odd on $(-p, p)$, then the Fourier series of f has only sine terms. Moreover

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{p}\right)$$

where

$$b_n = \frac{2}{p}\int_0^p f(x)\sin\left(\frac{n\pi x}{p}\right)\,dx.$$

Half Range Sine and Half Range Cosine Series

Suppose f is only defined for $0 < x < p$. We can **extend** f to the left, to the interval $(-p, 0)$, as either an even function or as an odd function. Then we can express f with **two distinct** series:

Half range cosine series $\quad f(x) = \dfrac{a_0}{2} + \displaystyle\sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{p}\right)$

where $\quad a_0 = \dfrac{2}{p}\displaystyle\int_0^p f(x)\,dx \quad$ and $\quad a_n = \dfrac{2}{p}\displaystyle\int_0^p f(x)\cos\left(\frac{n\pi x}{p}\right)\,dx.$

Half range sine series $\quad f(x) = \displaystyle\sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{p}\right)$

where $\quad b_n = \dfrac{2}{p}\displaystyle\int_0^p f(x)\sin\left(\frac{n\pi x}{p}\right)\,dx.$

Find the Half Range Sine Series of f

$$f(x) = 2 - x, \quad 0 < x < 2 \qquad p = 2$$

$$\frac{n\pi x}{p} = \frac{n\pi x}{2}$$

$$b_n = \frac{2}{2}\int_0^2 f(x)\sin\left(\frac{n\pi x}{2}\right)dx$$

$$= \int_0^2 (2-x)\sin\left(\frac{n\pi x}{2}\right)dx$$

Integrating by parts gives (up to an added constant)

$$\int (2-x)\sin\left(\frac{n\pi x}{2}\right)\,dx = -\frac{2(2-x)}{n\pi}\cos\left(\frac{n\pi x}{2}\right) - \frac{4}{n^2\pi^2}\sin\left(\frac{n\pi x}{2}\right)$$

$$= \frac{-2(2-x)}{n\pi}\cos\left(\frac{n\pi x}{2}\right) - \underbrace{\frac{4}{n^2\pi^2}\sin\left(\cancel{\frac{n\pi x}{2}}\right)}_{0}\Bigg|_0^2$$

$$= \frac{-2(2-2)}{n\pi}\cos\left(\frac{n\pi 2}{2}\right) - \frac{-2(2-0)}{n\pi}\cos(0)$$

$$= \frac{4}{n\pi}$$

The series is $f(x) = \sum_{n=1}^{\infty} \frac{4}{n\pi}\sin\left(\frac{n\pi x}{2}\right)$

Find the Half Range Cosine Series of f

$$f(x) = 2 - x, \quad 0 < x < 2$$

$$a_0 = \frac{2}{2}\int_0^2 f(x)\,dx = \int_0^2 (2-x)\,dx = 2x - \frac{x^2}{2}\Big|_0^2 = 4-2 = 2$$

$$a_n = \frac{2}{2}\int_0^2 f(x)\cos\left(\frac{n\pi x}{2}\right)dx = \int_0^2 (2-x)\cos\left(\frac{n\pi x}{2}\right)dx$$

Integrating by parts gives (up to an added constant)

$$\int (2-x)\cos\left(\frac{n\pi x}{2}\right)\,dx = \frac{2(2-x)}{n\pi}\sin\left(\frac{n\pi x}{2}\right) - \frac{4}{n^2\pi^2}\cos\left(\frac{n\pi x}{2}\right)$$

$$a_n = \frac{2(2-x)}{n\pi}\sin\left(\frac{n\pi x}{2}\right) - \frac{4}{n^2\pi^2}\cos\left(\frac{n\pi x}{2}\right)\Big|_0^2$$

$$= \frac{-4}{n^2\pi^2}\cos(n\pi) - \frac{-4}{n^2\pi^2}\cos(0)$$

$$= \frac{-4}{n^2\pi^2}\left((-1)^n - 1\right) = \frac{4}{n^2\pi^2}\left(1-(-1)^n\right)$$

So the cosine series is

$$f(x) = 1 + \sum_{n=1}^{\infty} \frac{4}{n^2\pi^2}\left(1-(-1)^n\right)\cos\left(\frac{n\pi x}{2}\right)$$

Example Continued...

We have two different half range series:

$$\text{Half range sine:} \quad f(x) = \sum_{n=1}^{\infty} \frac{4}{n\pi} \sin\left(\frac{n\pi x}{2}\right)$$

$$\text{Half range cosine:} \quad f(x) = 1 + \sum_{n=1}^{\infty} \frac{4(1-(-1)^n)}{n^2\pi^2} \cos\left(\frac{n\pi x}{2}\right).$$

We have two different series representations for this function each of which converge to $f(x)$ on the interval $(0, 2)$. The following plots show graphs of f along with partial sums of each of the series. When we plot over the interval $(-2, 2)$ we see the two different symmetries. Plotting over a larger interval such as $(-6, 6)$ we can see the periodic extensions of the two symmetries.

Plots of f with Half range series

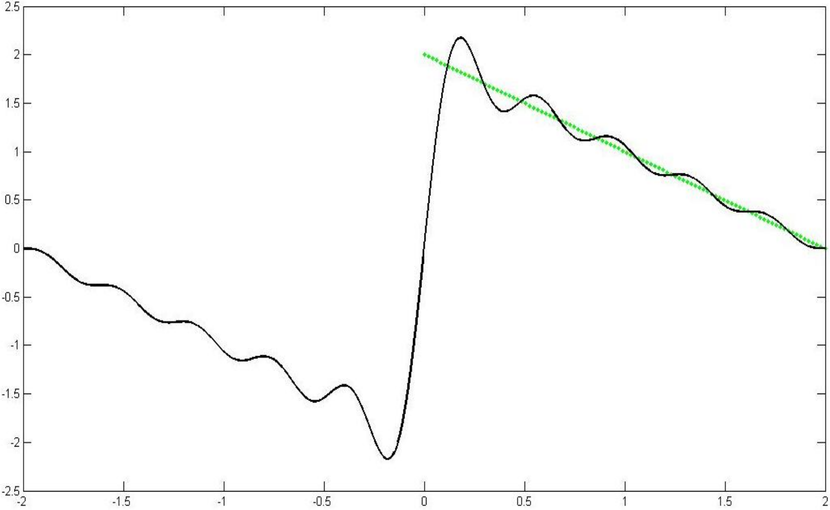

Figure: $f(x) = 2 - x$, $0 < x < 2$ with 10 terms of the sine series.

Plots of f with Half range series

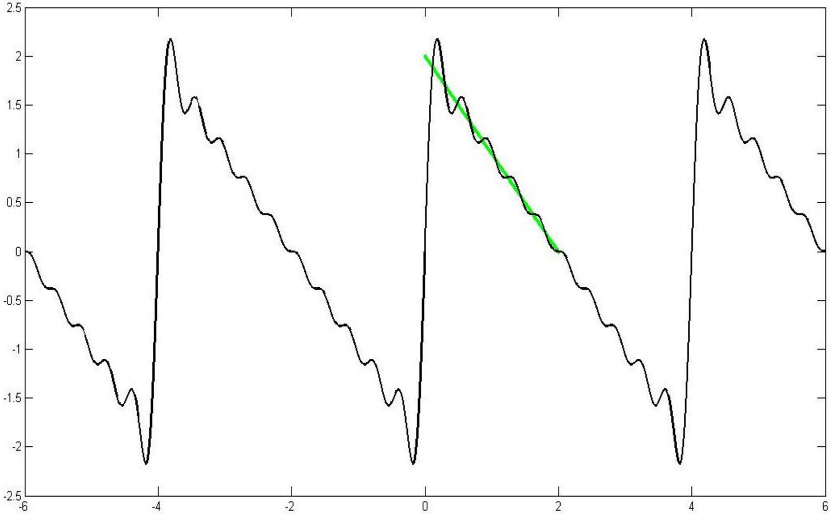

Figure: $f(x) = 2 - x$, $0 < x < 2$ with 10 terms of the sine series, and the series plotted over $(-6, 6)$

Plots of f with Half range series

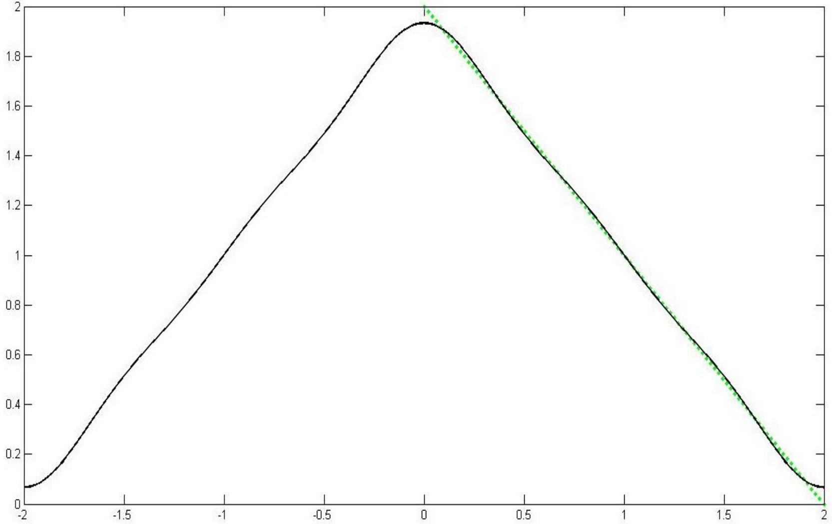

Figure: $f(x) = 2 - x$, $0 < x < 2$ with 5 terms of the cosine series.

Plots of f with Half range series

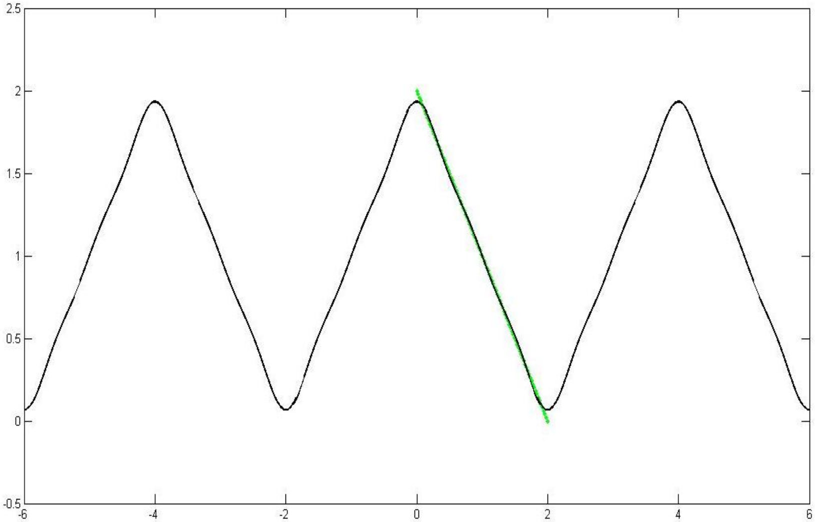

Figure: $f(x) = 2 - x$, $0 < x < 2$ with 5 terms of the cosine series, and the series plotted over $(-6, 6)$

Solution of a Differential Equation

An undamped spring mass system has a mass of 2 kg attached to a spring with spring constant 128 N/m. The mass is driven by an external force $f(t) = 2t$ for $-1 < t < 1$ that is 2-periodic so that $f(t+2) = f(t)$ for all $t > 0$. Determine a particular solution x_p for the displacement for $t > 0$.

The ODE is $mx'' + kx = f(t)$

$$2x'' + 128x = f(t)$$

Standard form $x'' + 64x = \frac{1}{2} f(t)$

We found last time that if $f(x) = x$ on $(-1, 1)$, then

$$f(x) = \sum_{n=1}^{\infty} \frac{2}{n\pi}(-1)^{n+1}\sin(n\pi x)$$

We can use this to express our function $f(t) = 2t$ on $(-1, 1)$ as a Fourier series

$$f(t) = 2\sum_{n=1}^{\infty} \frac{2}{n\pi}(-1)^{n+1}\sin(n\pi t)$$

Our ODE is

$$x'' + 64x = \sum_{n=1}^{\infty} \frac{2}{n\pi}(-1)^{n+1}\sin(n\pi t)$$

We'll use the method of undetermined coefficients

Assume
$$x_p = \sum_{n=1}^{\infty} \left(A_n \cos(n\pi t) + B_n \sin(n\pi t) \right)$$

It's left to the reader to show that $A_n = 0$ for all n. So we'll find the Bs.

$$x_p = \sum_{n=1}^{\infty} B_n \sin(n\pi t)$$

we need x_p''

$$x_p' = \sum_{n=1}^{\infty} B_n (n\pi) \cos(n\pi t)$$

$$x_p'' = \sum_{n=1}^{\infty} B_n \left(-(n\pi)^2 \sin(n\pi t) \right)$$

Plug into $\quad x_p'' + 64 x_p$

$$\sum_{n=1}^{\infty} -(n\pi)^2 B_n \sin(n\pi t) + 64 \sum_{n=1}^{\infty} B_n \sin(n\pi t) =$$

$$\sum_{n=1}^{\infty} \frac{2}{n\pi} (-1)^{n+1} \sin(n\pi t)$$

$$\sum_{n=1}^{\infty} \left[-(n\pi)^2 B_n \sin(n\pi t) + 64 B_n \sin(n\pi t) \right]$$

$$= \sum_{n=1}^{\infty} \frac{2}{n\pi} (-1)^{n+1} \sin(n\pi t)$$

$$\sum_{n=1}^{\infty} (64 - n^2\pi^2) B_n \sin(n\pi t) = \sum_{n=1}^{\infty} \frac{2}{n\pi} (-1)^{n+1} \sin(n\pi t)$$

Match Coefficients

For each $n \geq 1$

$$(64 - n^2\pi^2) B_n = \frac{2}{n\pi} (-1)^{n+1}$$

Since $64 - n^2\pi^2 \neq 0$ for all n

$$B_n = \frac{2}{n\pi(64-n^2\pi^2)}(-1)^{n+1}$$

So the particular solution

$$\boxed{x_p = \sum_{n=1}^{\infty} \frac{2}{n\pi(64-n^2\pi^2)}(-1)^{n+1}\sin(n\pi t)}$$