Section 8.1: The Laws of Sines and Cosines

In order to use the Law of Sines, we must know one angle-side pair (e.g. A and a). Since each angle is greater than 0° and less than 180°, all sine values are positive. So the law can be stated as

Law of Sines:

\[
\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}
\]

or

\[
\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}
\]

The Law of Sines can be used for AAS, ASA, or SSA.
Section 8.2: The Law of Cosines

**Theorem:** For the triangle labeled using the previous convention, all three of the following equations hold

\[
\begin{align*}
    a^2 &= b^2 + c^2 - 2bc \cos A \\
    b^2 &= a^2 + c^2 - 2ac \cos B \\
    c^2 &= a^2 + b^2 - 2ab \cos C
\end{align*}
\]

This can be used if two sides and an included angle (SAS) are known, or if the three sides (SSS) are known.
The Law of Cosines & The Pythagorean Theorem

Apply the law of cosines to a right triangle for which $C = 90^\circ$; see what it produces.

For hypotenuse $c$, and sides $a, b$

$$c^2 = a^2 + b^2 - 2ab \cos C$$

$$c^2 = a^2 + b^2 - 2ab \cos(90^\circ)$$

$$= a^2 + b^2 - 2ab \cdot 0$$

$$c^2 = a^2 + b^2$$

So the Pythagorean Thm is a special case of the Law of Cosines.
Example (SSS)

Solve the triangle given \( a = 5, \quad b = 2, \quad c = 6 \)

By the Law of Cosines

\[
\begin{align*}
    a^2 &= b^2 + c^2 - 2bc \cos A \\
    \sin^2 &= 2^2 + 6^2 - 2(2)(6) \cos A \\
    2S &= 40 - 24 \cos A \\
    -15 &= -24 \cos A \quad \Rightarrow \quad \cos A = \frac{15}{24} = \frac{5}{8} \\
    A &\approx 51.3^\circ \\
    c^2 &= a^2 + b^2 - 2ab \cos C
\end{align*}
\]
\[ \begin{align*}
6^2 &= a^2 + 5^2 - 2(2)(5) \cos C \\
36 &= 29 - 20 \cos C \\
7 &= -20 \cos C \quad \Rightarrow \quad \cos C = \frac{-7}{20} \\
C &\approx 110.5^\circ \\
B &= 180^\circ - A - C = 180^\circ - 51.3^\circ - 110.5^\circ = 18.2^\circ \\
A &= 51.3^\circ \quad B = 18.2^\circ \quad C = 110.5^\circ \\
a &= 5 \quad b = 2 \quad c = 6
\end{align*} \]
Area of a Triangle

**Theorem:** The area of a triangle is one half the product of any two sides times the sine of the included angle. That is

\[
\text{Area} = \frac{1}{2} ab \sin C = \frac{1}{2} ac \sin B = \frac{1}{2} bc \sin A.
\]

There is an alternative theorem that can be used if no angles are known.

**Theorem: (Heron’s Formula)** For the triangle with sides of lengths \(a\), \(b\), and \(c\). Define the semi-perimeter

\[
s = \frac{a + b + c}{2}.
\]

The area of the triangle is

\[
\text{Area} = \sqrt{s(s - a)(s - b)(s - c)}.
\]
Example

A set designer needs to estimate the amount of paint required to paint a triangular piece of wooden backdrop. Determine the area of the wood piece shown to the nearest tenth of a meter.
Use Heron's formula with \( a = 4 \), \( b = 7 \), \( c = 9 \)

\[
S = \frac{1}{2}(a+b+c) = \frac{1}{2}(4+7+9) = 10
\]

\[
S-a = 10-4 = 6
\]

\[
S-b = 10-7 = 3
\]

\[
S-c = 10-9 = 1
\]

\[
(Area)^2 = S(S-a)(S-b)(S-c) = 10(6)(3)(1) = 180
\]

So the area

\[
A = \sqrt{180} = 6\sqrt{5} \approx 13.4
\]

The area is approximately 13.4 m²