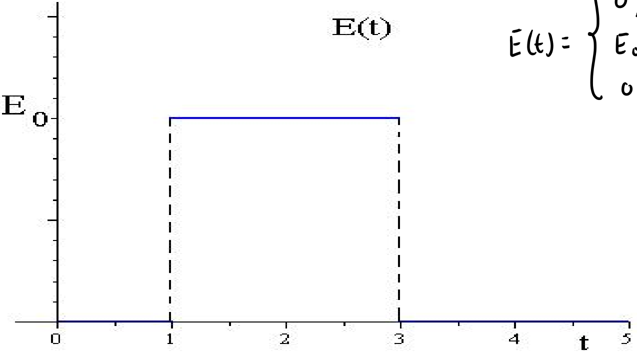
April 28 Math 2306 sec 59 Spring 2016

Section 16: Laplace Transforms of Derivatives and IVPs

An LR-series circuit has inductance $L = 1$h, resistance $R = 10\Omega$, and applied force $E(t)$ whose graph is given below. If the initial current $i(0) = 0$, find the current $i(t)$ in the circuit.



$$E(t) = \begin{cases} 0, & 0 \le t < 1 \\ E_0, & 1 \le t < 3 \\ 0, & t \ge 3 \end{cases}$$

LR Circuit Example

Last time, we determined that $E(t) = E_0\mathscr{U}(t-1) - E_0\mathscr{U}(t-3)$. This makes the IVP we're trying to solve

$$\frac{di}{dt} + 10i = E_0\mathscr{U}(t-1) - E_0\mathscr{U}(t-3), \quad i(0) = 0.$$

Let $\mathscr{L}\{i(t)\} = I(s)$

$$\mathscr{L}\{i' + 10i\} = \mathscr{L}\{E_0\mathscr{U}(t-1) - E_0\mathscr{U}(t-3)\}$$

$$\mathscr{L}\{i'\} + 10\mathscr{L}\{i\} = E_0\mathscr{L}\{\mathscr{U}(t-1)\} - E_0\mathscr{L}\{\mathscr{U}(t-3)\}$$

$$sI(s) - \underbrace{i(0)}_{=0} + 10I(s) = \frac{E_0}{s}e^{-s} - \frac{E_0}{s}e^{-3s}$$

$$(s+10)I(s) = \frac{E_0}{s}e^{-s} - \frac{E_0}{s}e^{-3s}$$

$$I(s) = \frac{E_0}{s(s+10)}e^{-s} - \frac{E_0}{s(s+10)}e^{-3s}$$

We need a partial fraction decomp on $\frac{1}{s(s+10)}$

$$\frac{1}{s(s+10)} = \frac{A}{s} + \frac{B}{s+10}$$

Mult. by $s(s+10)$

$$1 = A(s+10) + Bs$$

$$\underline{\underline{0}}s + \underline{1} = \underline{\underline{(A+B)}}s + \underline{10A}$$

$$10A=1 \Rightarrow A=\frac{1}{10} \qquad A+B=0 \Rightarrow B=-A=\frac{-1}{10}$$

$$I(s)=\left(\frac{\frac{1}{10}}{s}-\frac{\frac{1}{10}}{s+10}\right)E_0e^{-s}-\left(\frac{\frac{1}{10}}{s}-\frac{\frac{1}{10}}{s+10}\right)E_0e^{-3s}$$

$$=\frac{\frac{E_0}{10}}{s}e^{-s}-\frac{\frac{E_0}{10}}{s+10}e^{-s}-\frac{\frac{E_0}{10}}{s}e^{-3s}+\frac{\frac{E_0}{10}}{s+10}e^{-3s}$$

$$i(t)=\mathcal{L}^{-1}\left\{I(s)\right\}$$

$$=\frac{E_0}{10}\mathcal{L}^{-1}\left\{\frac{e^{-s}}{s}\right\}-\frac{E_0}{10}\mathcal{L}^{-1}\left\{\frac{e^{-s}}{s+10}\right\}-\frac{E_0}{10}\mathcal{L}^{-1}\left\{\frac{e^{-3s}}{s}\right\}+\frac{E_0}{10}\mathcal{L}^{-1}\left\{\frac{e^{-3s}}{s+10}\right\}$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s}\right\}=1 \quad \text{and} \quad \mathcal{L}^{-1}\left\{\frac{1}{s+10}\right\}=e^{-10t}$$

$$i(t)=\frac{E_0}{10}\mathcal{U}(t-1)-\frac{E_0}{10}e^{-10(t-1)}\mathcal{U}(t-1)-\frac{E_0}{10}\mathcal{U}(t-3)+\frac{E_0}{10}e^{-10(t-3)}\mathcal{U}(t-3)$$

Let's write this as a piecewise defined function in the usual form.

For $0\le t<1$, $t<3$ so $\mathcal{U}(t-1)=0$ and $\mathcal{U}(t-3)=0$

$$i(t)=0$$

For $1 \le t < 3$, $\mathcal{U}(t-1) = 1$ and $\mathcal{U}(t-3) = 0$

$$i(t) = \frac{E_0}{10} - \frac{E_0}{10} e^{-10(t-1)}$$

For $t \geq 3$, $t > 1$ $\mathcal{U}(t-1) = 1$ and $\mathcal{U}(t-3) = 1$

$$i(t) = \frac{E_0}{10} - \frac{E_0}{10} e^{-10(t-1)} - \frac{E_0}{10} + \frac{E_0}{10} e^{-10(t-3)}$$

$$= \frac{-E_0}{10} e^{-10(t-1)} + \frac{E_0}{10} e^{-10(t-3)}$$

$$= \frac{E_0}{10} \left(e^{-10(t-3)} - e^{-10(t-1)} \right)$$

$$
i(t) = \begin{cases} 0 & , \quad 0 \le t < 1 \\ \frac{E_0}{10} - \frac{E_0}{10} e^{-10(t-1)} & , \quad 1 \le t < 3 \\ \frac{E_0}{10}\left(e^{-10(t-3)} - e^{-10(t-1)} \right) & , \quad t \geq 3 \end{cases}
$$

Solve the IVP

$$y'' + y = \underbrace{\begin{cases} 3, & 0 \le t < 4 \\ 2t-5, & t \ge 4 \end{cases}}_{f(t)} \qquad y(0)=1, \quad y'(0)=0$$

$$f(t) = 3 - 3\mathcal{U}(t-4) + (2t-5)\mathcal{U}(t-4)$$

$$= 3 + (-3+2t-5)\mathcal{U}(t-4)$$

$$= 3 + (2t-8)\mathcal{U}(t-4)$$

$$= 3 + 2(t-4)\mathcal{U}(t-4)$$

* If $g(t) = t$
then $g(t-4) = t-4$

$$y'' + y = 3 + 2(t-4)\mathcal{U}(t-4), \quad y(0) = 1 \quad y'(0) = 0$$

Let $Y(s) = \mathcal{L}\{y(t)\}$

$$\mathcal{L}\{y'' + y\} = \mathcal{L}\{3 + 2(t-4)\mathcal{U}(t-4)\}$$

$$\mathcal{L}\{y''\} + \mathcal{L}\{y\} = 3\mathcal{L}\{1\} + 2\mathcal{L}\{(t-4)\mathcal{U}(t-4)\}$$

$$s^2Y(s) - s\underbrace{y(0)}_{=1} - \underbrace{y'(0)}_{=0} + Y(s) = \frac{3}{s} + \frac{2}{s^2}e^{-4s}$$

$$(s^2+1)Y(s) - s = \frac{3}{s} + \frac{2}{s^2}e^{-4s}$$

$$(s^2+1)Y(s) = \frac{3}{s} + \frac{2}{s^2}e^{-4s} + s$$

$$Y(s) = \frac{3}{s(s^2+1)} + \frac{2}{s^2(s^2+1)}e^{-4s} + \frac{s}{s^2+1}$$

We need a partial fraction decomp on $\frac{3}{s(s^2+1)}$ and $\frac{2}{s^2(s^2+1)}$

$$\frac{3}{s(s^2+1)} = \frac{A}{s} + \frac{Bs+C}{s^2+1}$$

Mult. by $s(s^2+1)$

$$3 = A(s^2+1) + (Bs+C)s$$

$$= As^2 + A + Bs^2 + Cs$$

$$0s^2 + 0s + 3 = (A+B)s^2 + Cs + A$$

$A=3, \quad C=0, \quad A+B=0 \Rightarrow B=-A=-3$

$$\frac{3}{s(s^2+1)} = \frac{3}{s} + \frac{-3s}{s^2+1}$$

$$\frac{2}{s^2(s^2+1)} = \frac{A}{s} + \frac{B}{s^2} + \frac{Cs+D}{s^2+1}$$

Mult. by $s^2(s^2+1)$

$$2 = As(s^2+1) + B(s^2+1) + (Cs+D)s^2$$

$$= As^3 + As + Bs^2 + B + Cs^3 + Ds^2$$

$$0s^3 + 0s^2 + 0s + 2 = (A+C)s^3 + (B+D)s^2 + As + B$$

$B=2$, $A=0$, $B+D=0 \Rightarrow D=-B=-2$, $A+C=0$ $C=0$

$$\frac{2}{s^2(s^2+1)} = \frac{2}{s^2} + \frac{-2}{s^2+1}$$

$$Y(s) = \frac{3}{s} - \frac{3s}{s^2+1} + \left(\frac{2}{s^2} - \frac{2}{s^2+1}\right)e^{-4s} + \frac{s}{s^2+1}$$

$$Y(s) = \frac{3}{s} - \frac{2s}{s^2+1} + \frac{2}{s^2}e^{-4s} - \frac{2}{s^2+1}e^{-4s}$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} = t \quad \text{and} \quad \mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\} = \sin t$$

$$y(t) = \mathcal{L}^{-1}\{Y(s)\}$$

$$= 3\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} - 2\mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\} + 2\mathcal{L}^{-1}\left\{\frac{e^{-4s}}{s^2}\right\} - 2\mathcal{L}^{-1}\left\{\frac{e^{-4s}}{s^2+1}\right\}$$

$$\boxed{y(t) = 3 - 2\cos t + 2(t-4)\mathcal{U}(t-4) - 2\sin(t-4)\mathcal{U}(t-4)}$$

The Second Shifting Theorem

Using the definition of the Laplace transform, show that

$$\mathscr{L}\{f(t-a)\mathscr{U}(t-a)\} = e^{-as}\mathscr{L}\{f(t)\}$$

Recall $\mathscr{U}(t-a) = \begin{cases} 0, & 0 \le t < a \\ 1, & t \geq a \end{cases}$

$$\mathscr{L}\{f(t-a)\mathscr{U}(t-a)\} = \int_0^\infty e^{-st} f(t-a)\mathscr{U}(t-a)\,dt$$

$$= \int_0^a e^{-st} f(t-a)\underbrace{\mathscr{U}(t-a)}_{=0}\,dt + \int_a^\infty e^{-st} f(t-a)\underbrace{\mathscr{U}(t-a)}_{=1}\,dt$$

$$= \int_a^\infty e^{-st} f(t-a)\, dt$$

Let $u = t - a$

$du = dt$

$u = t-a \Rightarrow t = u+a$

when $t = a$,

$u = a - a = 0$

as $t \to \infty$

$\lim_{t\to\infty} (t-a) = \infty$

$$= \int_0^\infty e^{-s(u+a)} f(u)\, du$$

$$= \int_0^\infty e^{-as} e^{-su} f(u)\, du$$

$$= e^{-as}\left(\int_0^\infty e^{-su} f(u)\, du\right)$$

$e^{-s(u+a)} = e^{-su} \cdot e^{-sa}$

$$= e^{-as}\mathcal{L}\{f(t)\}$$ as expected.