

Section 15: Shift Theorems

Theorem (translation in s)

Suppose $\mathcal{L}\{f(t)\} = F(s)$. Then for any real number a

$$\mathcal{L}\{e^{at}f(t)\} = F(s - a).$$

For example,

$$\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}} \implies \mathcal{L}\{e^{at}t^n\} = \frac{n!}{(s - a)^{n+1}}.$$

$$\mathcal{L}\{\cos(kt)\} = \frac{s}{s^2 + k^2} \implies \mathcal{L}\{e^{at}\cos(kt)\} = \frac{s - a}{(s - a)^2 + k^2}.$$

Inverse Laplace Transforms (completing the square)

(a) $\mathcal{L}^{-1} \left\{ \frac{s}{s^2 + 2s + 2} \right\}$ $s^2 + 2s + 2$ is irreducible
(doesn't factor)

Complete the square

$$\begin{aligned} s^2 + 2s + 2 &= s^2 + 2s + 1 - 1 + 2 \\ &= (s+1)^2 + 1 \end{aligned}$$

$$\frac{s}{s^2 + 2s + 2} = \frac{s}{(s+1)^2 + 1}$$

← this needs to be $s+1$ as well

Use $s = s+1 - 1$

$$\frac{s}{s^2 + 2s + 2} = \frac{s+1-1}{(s+1)^2 + 1} = \frac{s+1}{(s+1)^2 + 1} - \frac{1}{(s+1)^2 + 1}$$

Use $\mathcal{L}\{\cos t\} = \frac{s}{s^2 + 1}$ and $\mathcal{L}\{\sin t\} = \frac{1}{s^2 + 1}$

and if $s-a = s+1$, then $a = -1$

$$\begin{aligned}\mathcal{L}^{-1}\left\{\frac{s}{s^2 + 2s + 2}\right\} &= \mathcal{L}^{-1}\left\{\frac{s+1}{(s+1)^2 + 1} - \frac{1}{(s+1)^2 + 1}\right\} \\ &= \mathcal{L}^{-1}\left\{\frac{s+1}{(s+1)^2 + 1}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2 + 1}\right\} \\ &= e^{-t} \cos t - e^{-t} \sin t\end{aligned}$$

Inverse Laplace Transforms (repeat linear factors)

(b) $\mathcal{L}^{-1}\left\{\frac{1+3s-s^2}{s(s-1)^2}\right\}$ We'll do a partial fraction decomp

$$\frac{-s^2+3s+1}{s(s-1)^2} = \frac{A}{s} + \frac{B}{s-1} + \frac{C}{(s-1)^2}$$

Clear fractions

$$-s^2+3s+1 = A(s-1)^2 + Bs(s-1) + Cs$$

$$= A(s^2-2s+1) + B(s^2-s) + Cs$$

$$\underline{-s^2} + \underline{3s} + \underline{1} = \underline{(A+B)}s^2 + \underline{(-2A-B+C)}s + \underline{A}$$

Matching coefficients

$$\begin{array}{rcl} A + B & = & -1 \\ -2A - B + C & = & 3 \\ A & = & 1 \end{array} \quad \left. \begin{array}{l} \leftarrow \\ \leftarrow \\ \leftarrow \end{array} \right\} B = -1 - A = -1 - 1 = -2$$

$$\text{Then } C = 3 + 2A + B = 3 + 2(1) - 2 = 3$$

$$\frac{-s^2 + 3s + 1}{s(s-1)^2} = \frac{1}{s} - \frac{2}{s-1} + \frac{3}{(s-1)^2}$$

$$\mathcal{L}^{-1}\left\{\frac{3}{s^2}\right\} = 3 \mathcal{L}^{-1}\left\{\frac{1!}{s^2}\right\} = 3t \quad \text{and } s-1 = s-a \Rightarrow a=1$$

$$\mathcal{L}^{-1}\left\{\frac{-s^2 + 3s + 1}{s(s-1)^2}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s}\right\} - 2\mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\} + 3\mathcal{L}^{-1}\left\{\frac{1}{(s-1)^2}\right\}$$
$$= 1 - 2e^t + 3e^t t$$

The Unit Step Function

Let $a > 0$. The unit step function $\mathcal{U}(t - a)$ is defined by

$$\mathcal{U}(t - a) = \begin{cases} 0, & 0 \leq t < a \\ 1, & t \geq a \end{cases}$$

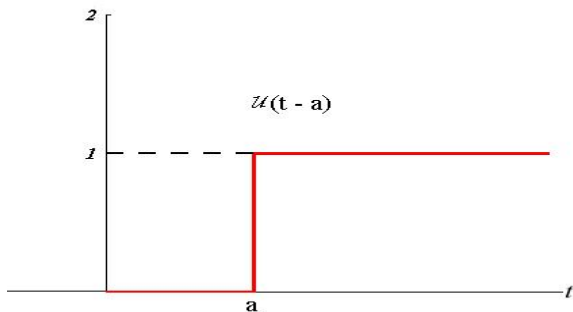


Figure: We can use the unit step function to provide convenient expressions for piecewise defined functions.

Piecewise Defined Functions

Verify that $f(t) = \begin{cases} g(t), & 0 \leq t < a \\ h(t), & t \geq a \end{cases}$ for some positive a

$$f(t) = \begin{cases} g(t), & 0 \leq t < a \\ h(t), & t \geq a \end{cases} = g(t) - g(t)\mathcal{U}(t-a) + h(t)\mathcal{U}(t-a)$$

We'll show that the far right expression is equal to f on the two intervals $[0, a)$ and $[a, \infty)$.

If $0 \leq t < a$, then $\mathcal{U}(t-a) = 0$. On this interval

$$\begin{aligned} g(t) - g(t)\mathcal{U}(t-a) + h(t)\mathcal{U}(t-a) &= g(t) - g(t) \cdot 0 + h(t) \cdot 0 \\ &= g(t) \end{aligned}$$

For $t \geq a$, $u(t-a) = 1$. On this interval

$$\begin{aligned}g(t) - g(t)u(t-a) + h(t)u(t-a) &= g(t) - g(t) \cdot 1 + h(t) \cdot 1 \\ &= h(t)\end{aligned}$$

So $g(t) - g(t)u(t-a) + h(t)u(t-a) = f(t)$

for all $t \geq 0$.

Piecewise Defined Functions in Terms of \mathcal{U}

Write f on one line in terms of \mathcal{U} as needed

$$f(t) = \begin{cases} e^t, & 0 \leq t < 2 \\ t^2, & 2 \leq t < 5 \\ 2t & t \geq 5 \end{cases}$$

We can use unit step functions to turn the pieces
"on" and "off."

$$f(t) = e^t - e^t u(t-2) + t^2 u(t-2) - t^2 u(t-5) + 2t u(t-5)$$

\uparrow on \uparrow off \uparrow on \uparrow off \uparrow on

Verification left as an exercise

Translation in t

Given a function $f(t)$ for $t \geq 0$, and a number $a > 0$

$$f(t-a)\mathcal{U}(t-a) = \begin{cases} 0, & 0 \leq t < a \\ f(t-a), & t \geq a \end{cases}.$$

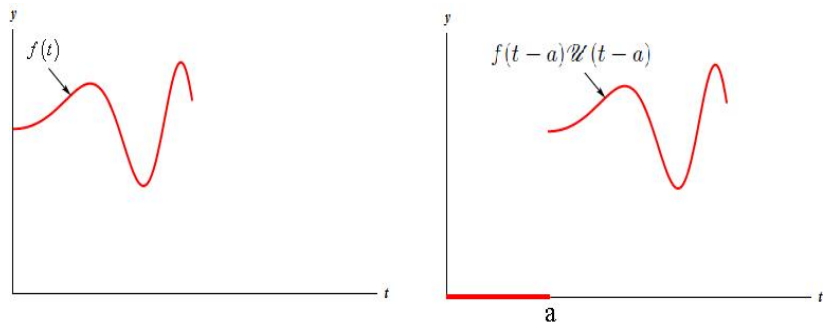


Figure: The function $f(t-a)\mathcal{U}(t-a)$ has the graph of f shifted a units to the right with value of zero for t to the left of a .

Theorem (translation in t)

If $F(s) = \mathcal{L}\{f(t)\}$ and $a > 0$, then

$$\mathcal{L}\{f(t-a)\mathcal{U}(t-a)\} = e^{-as}F(s).$$

In particular,

← the $f(t) = 1$ here

$$\mathcal{L}\{\mathcal{U}(t-a)\} = \frac{e^{-as}}{s}.$$

As another example,

$$\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}} \implies \mathcal{L}\{(t-a)^n\mathcal{U}(t-a)\} = \frac{n!e^{-as}}{s^{n+1}}.$$

Find $\mathcal{L}\{u(t-a)\}$ for some $a > 0$

$$u(t-a) = \begin{cases} 0, & 0 \leq t < a \\ 1, & t \geq a \end{cases}$$

By definition

$$\mathcal{L}\{u(t-a)\} = \int_0^{\infty} e^{-st} u(t-a) dt$$

$$= \int_0^a e^{-st} \cdot 0 dt + \int_a^{\infty} e^{-st} \cdot 1 dt$$

$$= \left. \frac{-1}{s} e^{-st} \right|_a^{\infty} \quad \text{for } s > 0$$

$$= \frac{-1}{s} (0 - e^{-as}) = \frac{e^{-as}}{s}$$

Example

Find the Laplace transform $\mathcal{L}\{f(t)\}$ where

$$f(t) = \begin{cases} 1, & 0 \leq t < 1 \\ t, & t \geq 1 \end{cases}$$

(a) First write f in terms of unit step functions.

$$f(t) = 1 - 1u(t-1) + tu(t-1)$$

$$= 1 + (-1 + t)u(t-1)$$

$$= 1 + (t-1)u(t-1)$$

Example Continued...

(b) Now use the fact that $f(t) = 1 + (t - 1)\mathcal{U}(t - 1)$ to find $\mathcal{L}\{f\}$.

$$\begin{aligned}\mathcal{L}\{f(t)\} &= \mathcal{L}\{1 + (t-1)\mathcal{U}(t-1)\} \\ &= \mathcal{L}\{1\} + \mathcal{L}\{(t-1)\mathcal{U}(t-1)\} \\ &= \frac{1}{s} + e^{-1s} \left(\frac{1}{s^2} \right) = \frac{1}{s} + \frac{e^{-s}}{s^2}\end{aligned}$$

Note: If $g(t) = t$ then $g(t-1) = t-1$

and $\mathcal{L}\{t\} = \frac{1}{s^2}$ If $t-a = t-1$ then $a=1$