April 8 Math 2306 sec. 60 Spring 2019

Section 15: Shift Theorems

Theorem (translation in s)
Suppose $\mathscr{L}\{f(t)\} = F(s)$. Then for any real number a

$$\mathscr{L}\left\{e^{at} f(t)\right\} = F(s-a).$$

For example,

$$\mathscr{L}\left\{t^n\right\} = \frac{n!}{s^{n+1}} \quad \Longrightarrow \quad \mathscr{L}\left\{e^{at} t^n\right\} = \frac{n!}{(s-a)^{n+1}}.$$

$$\mathscr{L}\left\{\cos(kt)\right\} = \frac{s}{s^2+k^2} \quad \Longrightarrow \quad \mathscr{L}\left\{e^{at} \cos(kt)\right\} = \frac{s-a}{(s-a)^2+k^2}.$$

Inverse Laplace Transforms (completing the square)

(a) $\mathscr{L}^{-1}\left\{\dfrac{s}{s^2+2s+2}\right\}$

s^2+2s+2 is irreducible
(doesn't factor)

we'll complete the square

$$s^2+2s+2 = s^2+2s+1-1+2$$
$$= (s+1)^2+1$$

$$\frac{s}{s^2+2s+2} = \frac{s}{(s+1)^2+1}$$

← we need s+1 here too

we'll use $s = s+1-1$

So $$\frac{s}{s^2+2s+2} = \frac{s+1-1}{(s+1)^2+1} = \frac{s+1}{(s+1)^2+1} - \frac{1}{(s+1)^2+1}$$

Note $\mathcal{L}\{\cos t\} = \dfrac{s}{s^2+1}$ and $\mathcal{L}\{\sin t\} = \dfrac{1}{s^2+1}$

From $s+1 = s-a$, we have $a=-1$

$$\mathcal{L}^{-1}\left\{\frac{s}{s^2+2s+2}\right\} = \mathcal{L}^{-1}\left\{\frac{s+1}{(s+1)^2+1}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2+1}\right\}$$

$$= e^{-t}\cos t - e^{-t}\sin t$$

Inverse Laplace Transforms (repeat linear factors)

(b) $\mathscr{L}^{-1}\left\{\dfrac{1+3s-s^2}{s(s-1)^2}\right\}$ we'll do a partial fraction decomp

$$\frac{-s^2+3s+1}{s(s-1)^2} = \frac{A}{s} + \frac{B}{s-1} + \frac{C}{(s-1)^2}$$

Clear fractions

$$-s^2+3s+1 = A(s-1)^2 + Bs(s-1) + Cs$$

$$= A(s^2-2s+1) + B(s^2-s) + Cs$$

$$-s^2+3s+1 = (A+B)s^2 + (-2A-B+C)s + A$$

$$\left.\begin{aligned} A + B &= -1 \\ -2A - B + C &= 3 \\ A &= 1 \end{aligned}\right\} \Rightarrow \quad B = -1 - A = -1 - 1 = -2$$

and

$$C = 3 + 2A + B = 3 + 2 - 2 = 3$$

So

$$\frac{-s^2 + 3s + 1}{s(s-1)^2} = \frac{1}{s} - \frac{2}{s-1} + \frac{3}{(s-1)^2}$$

Note that $\mathcal{L}\{t\} = \frac{1}{s^2}$ and if $s - a = s - 1$

then $a = 1$

Then
$$\mathscr{L}^{-1}\left\{\frac{-s^2+3s+1}{s(s-1)^2}\right\} = \mathscr{L}^{-1}\left\{\frac{1}{s}\right\} - 2\mathscr{L}^{-1}\left\{\frac{1}{s-1}\right\} + 3\mathscr{L}^{-1}\left\{\frac{1}{(s-1)^2}\right\}$$

$$= 1 - 2e^t + 3e^t t$$

The Unit Step Function

Let $a \geq 0$. The unit step function $\mathscr{U}(t-a)$ is defined by

$$\mathscr{U}(t-a) = \begin{cases} 0, & 0 \leq t < a \\ 1, & t \geq a \end{cases}$$

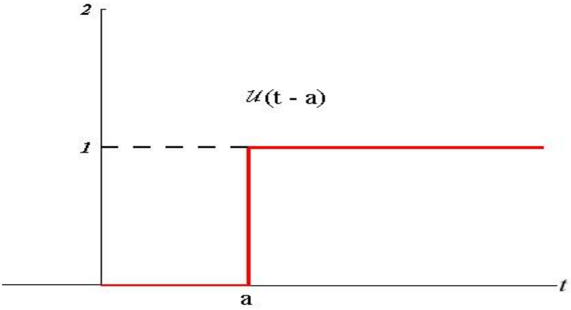

Figure: We can use the unit step function to provide convenient expressions for piecewise defined functions.

Piecewise Defined Functions

Verify that

$$f(t) = \begin{cases} g(t), & 0 \le t < a \\ h(t), & t \ge a \end{cases} = g(t) - g(t)\mathscr{U}(t-a) + h(t)\mathscr{U}(t-a)$$

We'll consider how f is given for $0 \le t < a$ and for $t \ge a$. Working with the far right.

If $0 \le t < a$, then $\mathscr{U}(t-a) = 0$. On this interval

$$g(t) - g(t)\mathscr{U}(t-a) + h(t)\mathscr{U}(t-a) = g(t) - g(t)\cdot 0 + h(t)\cdot 0$$
$$= g(t)$$

If $t \geq a$, then $\mathcal{U}(t-a) = 1$. On this interval

$$g(t) - g(t)\mathcal{U}(t-a) + h(t)\mathcal{U}(t-a) = g(t) - g(t)\cdot 1 + h(t)\cdot 1$$
$$= h(t)$$

So $g(t) - g(t)\mathcal{U}(t-a) + h(t)\mathcal{U}(t-a) = f(t)$

on all of $[0, \infty)$.

Piecewise Defined Functions in Terms of $\mathscr{U}$

Write f on one line in terms of $\mathscr{U}$ as needed

$$f(t) = \begin{cases} e^t, & 0 \le t < 2 \\ t^2, & 2 \le t < 5 \\ 2t & t \ge 5 \end{cases}$$

We can turn the pieces "on" and "off" using $\mathscr{U}(t-a)$

$$f(t) = \underbrace{e^t}_{\text{on}} - \underbrace{e^t\mathscr{U}(t-2)}_{\text{off}} + \underbrace{t^2\mathscr{U}(t-2)}_{\text{on}} - \underbrace{t^2\mathscr{U}(t-5)}_{\text{off}} + \underbrace{2t\,\mathscr{U}(t-5)}_{\text{on}}$$

Verification left as an exercise.

Translation in t

Given a function $f(t)$ for $t \geq 0$, and a number $a > 0$

$$f(t-a)\mathscr{U}(t-a) = \begin{cases} 0, & 0 \leq t < a \\ f(t-a), & t \geq a \end{cases}.$$

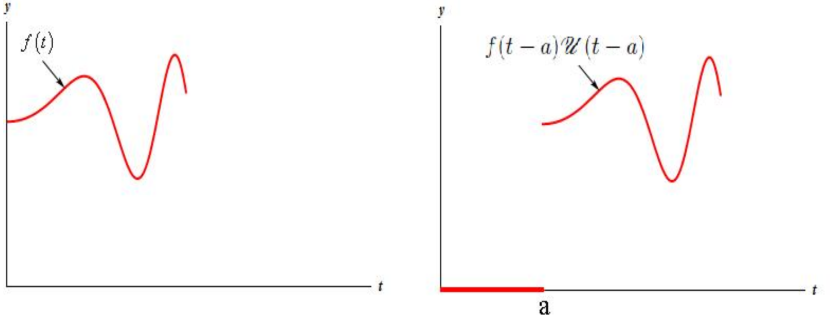

Figure: The function $f(t-a)\mathscr{U}(t-a)$ has the graph of f shifted a units to the right with value of zero for t to the left of a.

Theorem (translation in t)

If $F(s) = \mathscr{L}\{f(t)\}$ and $a > 0$, then

$$\mathscr{L}\{f(t-a)\mathscr{U}(t-a)\} = e^{-as}F(s).$$

In particular,

$$\mathscr{L}\{\mathscr{U}(t-a)\} = \frac{e^{-as}}{s}.$$

As another example,

$$\mathscr{L}\{t^n\} = \frac{n!}{s^{n+1}} \quad \Longrightarrow \quad \mathscr{L}\{(t-a)^n\mathscr{U}(t-a)\} = \frac{n!e^{-as}}{s^{n+1}}.$$

Find $\mathscr{L}\{\mathscr{U}(t-a)\}$

$$\mathcal{U}(t-a) = \begin{cases} 0, & 0 \le t < a \\ 1, & t \ge a \end{cases}$$

By definition

$$\mathscr{L}\{\mathcal{U}(t-a)\} = \int_0^\infty e^{-st}\,\mathcal{U}(t-a)\,dt$$

$$= \int_0^a e^{-st}\cdot 0\,dt + \int_a^\infty e^{-st}\cdot 1\,dt$$

$$= \frac{-1}{s}e^{-st}\Big|_a^\infty \quad \text{for } s>0$$

$$= \frac{-1}{s}\left(0 - e^{-s(a)}\right) = \frac{e^{-as}}{s}$$

Example

Find the Laplace transform $\mathscr{L}\left\{f(t)\right\}$ where

$$f(t) = \begin{cases} 1, & 0 \leq t < 1 \\ t, & t \geq 1 \end{cases}$$

(a) First write f in terms of unit step functions.

$$f(t) = 1 - 1\mathcal{U}(t-1) + t\,\mathcal{U}(t-1)$$

$$= 1 + (-1+t)\mathcal{U}(t-1)$$

$$= 1 + (t-1)\mathcal{U}(t-1)$$

Example Continued...

(b) Now use the fact that $f(t) = 1 + (t-1)\mathscr{U}(t-1)$ to find $\mathscr{L}\{f\}$.

$$\mathscr{L}\{f(t)\} = \mathscr{L}\{1 + (t-1)\mathscr{U}(t-1)\}$$

$$= \mathscr{L}\{1\} + \mathscr{L}\{(t-1)\mathscr{U}(t-1)\}$$

$$= \frac{1}{s} + \frac{e^{-s}}{s^2}$$

$*$ $\mathscr{L}\{t\} = \frac{1}{s^2}$ so $\mathscr{L}\{(t-1)\mathscr{U}(t-1)\} = e^{-s}\left(\frac{1}{s^2}\right)$

A Couple of Useful Results

Another formulation of this translation theorem is

$$(1) \quad \mathscr{L}\{g(t)\mathscr{U}(t-a)\} = e^{-as}\mathscr{L}\{g(t+a)\}.$$

Because $g(t) = g((t-a)+a)$

Example: Find $\mathscr{L}\left\{\cos t\,\mathscr{U}\left(t-\frac{\pi}{2}\right)\right\}$