August 15 Math 2306 sec. 53 Fall 2018

Section 1: Concepts and Terminology

We have defined *differential equations*, and started to define certain characteristics and categories:

- Ordinary differential equations (ODEs) have one independent variable; partial differential equations (PDEs) have two or more independent variables.
- The **order** of an equation is equal to largest order of derivative appearing in the equation.
- An n^{th} order equation in *normal form* looks like $\dfrac{d^n y}{dx^n} = f(x, y, y', \ldots, y^{(n-1)})$ for some function f.

Classifications

Linearity: An n^{th} order differential equation is said to be **linear** if it can be written in the form

$$a_n(x)\frac{d^ny}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x).$$

The key characteristics here are:

- y and any of its derivatives can only appear as themselves (to the first power),
- coefficients of y and its derivatives may depend on the **independent** variable, but not on y or its derivatives,
- if you replace y on the left side with Cy for any constant C, it factors giving you C times the original left side,
- if you replace y on the left with $y_1 + y_2$ (a sum of y's), you can distribute to get a sum of the left side with y_1 and the same left side with y_2.

Classifications

Linearity: An n^{th} order differential equation is said to be **linear** if it can be written in the form

$$a_n(x)\frac{d^ny}{dx^n}+a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}}+\cdots+a_1(x)\frac{dy}{dx}+a_0(x)y=g(x).$$

If we define the operation L by

$$Ly=a_n(x)\frac{d^ny}{dx^n}+a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}}+\cdots+a_1(x)\frac{dy}{dx}+a_0(x)y$$

then L is a **linear operator** in the sense that

$$L(Cy)=CLy \quad \text{for any constant } C, \quad \text{and}$$

$$L(y_1+y_2)=Ly_1+Ly_2,$$

for any pair of sufficiently differentiable functions y_1 and y_2.

Examples (Linear -vs- Nonlinear)

$$a_n(x)\frac{d^ny}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x).$$

The following are linear.

$$y'' + 4y = 0$$

$a_2(x)y'' + a_1(x)y' + a_0(x)y = g(x)$

$a_2(x) = 1$

$a_1(x) = 0$

$a_0(x) = 4$

$g(x) = 0$

$$t^2\frac{d^2x}{dt^2} + 2t\frac{dx}{dt} - x = e^t$$

$a_2(t)x'' + a_1(t)x' + a_0(t)x = g(t)$

$a_2(t) = t^2$

$a_1(t) = 2t$

$a_0(t) = -1$

$g(t) = e^t$

Examples (Linear -vs- Nonlinear)

$$a_n(x)\frac{d^ny}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x).$$

The following are nonlinear.

$$\frac{d^3y}{dx^3} + \left(\frac{dy}{dx}\right)^4 = x^3$$

$$y''' + (y')^3 y' = x^3$$

$\left(\frac{dy}{dx}\right)^4$ is a nonlinear term

$$u'' + u' = \cos u$$

Cos u is a nonlinear term because u is dependent.

Example: Classification

Identify the independent and dependent variables. Determine the order of the equation. State whether it is linear or nonlinear.

(a) $y'' + 2ty' = \cos t + y - y'''$

$y''' + y'' + 2ty' - y = \cos t$

y - dependent

t - independent

order is 3

Linear

(b) $\ddot{\theta} + \frac{g}{\ell}\sin\theta = 0$ $\quad g$ and ℓ are constant

Independent variable is time (t)

θ - dependent variable

2^{nd} order

Non linear $\sin\theta$ is a nonlinear term

θ is dependent

Solution of $F(x, y, y', \ldots, y^{(n)}) = 0$ (*)

Definition: A function ϕ defined on an interval[1] I and possessing at least n continuous derivatives on I is a **solution** of (*) on I if upon substitution (i.e. setting $y = \phi(x)$) the equation reduces to an identity.

Definition: An **implicit solution** of (*) is a relation $G(x, y) = 0$ provided there exists at least one function $y = \phi$ that satisfies both the differential equation (*) and this relation.

[1] The interval is called the *domain of the solution* or the *interval of definition*.

Examples:

Verify that the given function is a solution of the ODE on the indicated interval.

$$\phi(t) = 3e^{2t}, \quad I = (-\infty, \infty), \quad \frac{d^2y}{dt^2} - \frac{dy}{dt} - 2y = 0$$

ϕ has derivatives of all orders on I.

In particular, it has at least two derivatives on I.

To finish the verification, we set $y = \phi(t)$

$y' = \phi'(t) = 6e^{2t}$, $y'' = \phi''(t) = 12e^{2t}$

$y'' - y' - 2y = 0$

$$12e^{2t} - 6e^{2t} - 2(3e^{2t}) =$$

$$12e^{2t} - 6e^{2t} - 6e^{2t} =$$

$$(12-6-6)e^{2t} =$$

$$0 = 0 \quad \text{an identity!}$$

So ϕ is a solution on I.

Verify that the relation(left) defines and implicit solution of the differential equation (right).

$$y^2 - 2x^2 y = 1, \qquad \frac{dy}{dx} = \frac{2xy}{y - x^2}$$

we can show that if y satisfies the relation, then it satisfies the ODE using implicit differentiation.

$$y^2 - 2x^2 y = 1 \Rightarrow 2y\frac{dy}{dx} - 4xy - 2x^2\frac{dy}{dx} = 0$$

Using some algebra

It may not be possible to clearly identify the domain of definition of an implicit solution.

$$y\frac{dy}{dx} - 2xy - x^2\frac{dy}{dx} = 0$$

$$(y-x^2)\frac{dy}{dx} = 2xy \qquad \text{for } y-x^2 \neq 0$$

$$\frac{dy}{dx} = \frac{2xy}{y-x^2} \qquad \text{this is the ODE}$$

Hence the relation defines an implicit solution.