August 15 Math 2306 sec. 56 Fall 2017

Section 1: Concepts and Terminology

Suppose $y = \phi(x)$ is a differentiable function. We know that $dy/dx = \phi'(x)$ is another (related) function.

For example, if $y = \cos(2x)$, then y is differentiable on $(-\infty, \infty)$. In fact,

$$\frac{dy}{dx} = -2\sin(2x).$$

Even dy/dx is differentiable with $d^2y/dx^2 = -4\cos(2x)$.

Suppose $y = \cos(2x)$

Note that $$\frac{d^2y}{dx^2} + 4y = 0.$$

$\frac{d^2y}{dx^2} = -4\cos(2x)$ and $4y = 4\cos(2x)$

So yes, $\frac{d^2y}{dx^2} + 4y = -4\cos(2x) + 4\cos(2x)$

$= 0$

A *differential* equation

The equation

$$\frac{d^2y}{dx^2} + 4y = 0.$$

is an example of a **differential equation.**

Questions: If we only started with the equation, how could we determine that $\cos(2x)$ satisfies it? Also, is $\cos(2x)$ the only possible function that y could be?

we'll be able to answer both at the end of this term!

Definition

A **Differential Equation** is an equation containing the derivative(s) of one or more dependent variables, with respect to one or more indendent variables.

Solving a differential equation refers to determining the dependent variable(s)—as function(s).

Independent Variable: will appear as one that derivatives are taken with respect to.

Dependent Variable: will appear as one that derivatives are taken of.

↑ these are functions

Independent and Dependent Variables

Often, the derivatives indicate which variable is which:

$$\frac{dy}{dx} \qquad \frac{du}{dt} \qquad \frac{dx}{dr}$$

$du \leftarrow$ dep. $\quad dt \leftarrow$ ind.

$dx \leftarrow$ dep $\quad dr \leftarrow$ ind.

derivative of y with respect to x

y - dependent

x - independent

Classifications

Type: An **ordinary differential equation (ODE)** has exactly one independent variable[1]. For example

$$\frac{dy}{dx} - y^2 = 3x, \quad \text{or} \quad \frac{dy}{dt} + 2\frac{dx}{dt} = t, \quad \text{or} \quad y'' + 4y = 0$$

A **partial differential equation (PDE)** has two or more independent variables. For example

$$\frac{\partial y}{\partial t} = \frac{\partial^2 y}{\partial x^2}, \quad \text{or} \quad \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0$$

[1] These are the subject of this course.

Classifications

Order: The order of a differential equation is the same as the highest order derivative appearing anywhere in the equation.

$$\frac{dy}{dx} - y^2 = 3x$$ 1st order

$$y''' + (y')^4 = x^3$$ 3rd order

$$\frac{\partial y}{\partial t} = \frac{\partial^2 y}{\partial x^2}$$ 2nd order

Notations and Symbols

We'll use standard derivative notations:

$$\text{Leibniz:} \quad \frac{dy}{dx}, \quad \frac{d^2y}{dx^2}, \quad \ldots \quad \frac{d^ny}{dx^n}, \quad \text{or}$$

$$\text{Prime \& superscripts:} \quad y', \quad y'', \quad \ldots \quad y^{(n)}.$$

Newton's **dot notation** may be used if the independent variable is time. For example if s is a position function, then

$$\text{velocity is } \frac{ds}{dt} = \dot{s}, \quad \text{and acceleration is } \frac{d^2s}{dt^2} = \ddot{s}$$

Notations and Symbols

An n^{th} order ODE, with independent variable x and dependent variable y can always be expressed as an equation

$$F(x, y, y', \dots, y^{(n)}) = 0$$

where F is some real valued function of $n + 2$ variables.

Normal Form: If it is possible to isolate the highest derivative term, then we can write a **normal form** of the equation

$$\frac{d^n y}{dx^n} = f(x, y, y', \dots, y^{(n-1)}).$$

$$\frac{d^2y}{dx^2} + 4y = 0$$

This has the form $F(x, y, y', y'') = 0$

where $F(x, y, y', y'') = y'' + 4y$

In normal form, this is

$$\frac{d^2y}{dx^2} = -4y$$

here $f(x, y, y') = -4y$

Notations and Symbols

If $n = 1$ or $n = 2$, an equation in normal form would look like

$$\frac{dy}{dx} = f(x,y) \quad \text{or} \quad \frac{d^2y}{dx^2} = f(x,y,y').$$

Differential Form: A first order equation may appear in the form

$$\underbrace{M(x,y)\,dx + N(x,y)\,dy}_{\text{differential form}} = 0$$

$$M(x,y)\,dx + N(x,y)\,dy = 0$$

Differential forms may be written in normal form in a couple of ways.

If $N(x,y) \neq 0$ then $N(x,y)\,dy = -M(x,y)\,dx$

$$\Rightarrow \frac{dy}{dx} = \frac{-M(x,y)}{N(x,y)}$$

If $M(x,y) \neq 0$ we can also write this

as $$\frac{dx}{dy} = \frac{-N(x,y)}{M(x,y)}$$

Classifications

Linearity: An n^{th} order differential equation is said to be **linear** if it can be written in the form

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x).$$

Note that each of the coefficients $a_0, \ldots, a_n$ and the right hand side g may depend on the independent variable but not on the dependent variable or any of its derivatives.

$y, y', \ldots, y^{(n)}$ can only appear to the 1st power and not inside any function or multiplied together.

Examples (Linear -vs- Nonlinear)

Both linear

$$y'' + 4y = 0$$

$a_2(x) = 1$

$a_1(x) = 0$

$a_0(x) = 4$

$g(x) = 0$

$$t^2\frac{d^2x}{dt^2} + 2t\frac{dx}{dt} - x = e^t$$

$a_2(t) = t^2$

$a_1(t) = 2t$

$a_0(t) = -1$

$g(t) = e^t$

Both nonlinear

$$\frac{d^3y}{dx^3} + \underbrace{\left(\frac{dy}{dx}\right)^4}_{\text{non linear term}} = x^3$$

$$u'' + u' = \cos u$$

nonlinear term

dependent variable inside the cosine.

Identify the independent and dependent variables. Determine the order of the equation. State whether it is linear or nonlinear.

(a) $y'' + 2ty' = \cos t + y - y'''$

It's 3^{rd} order

dependent $- y$

independent $- t$

rearrange

$y''' + y'' + 2ty' - y = \cos t$

This is linear.

(b) $\ddot{\theta}+\dfrac{g}{\ell}\sin\theta = 0$ $\quad g$ and ℓ are constant

2nd order

Independent variable is t for time

dependent - θ

Nonlinear due to the $\sin\theta$ term

as θ is dependent

Solution of $F(x, y, y', \ldots, y^{(n)}) = 0$ (*)

Definition: A function ϕ defined on an interval[2] I and possessing at least n continuous derivatives on I is a **solution** of (*) on I if upon substitution (i.e. setting $y = \phi(x)$) the equation reduces to an identity.

Definition: An **implicit solution** of (*) is a relation $G(x, y) = 0$ provided there exists at least one function $y = \phi$ that satisfies both the differential equation (*) and this relation.

[2] The interval is called the *domain of the solution* or the *interval of definition*.

Examples:

Verify that the given function is an solution of the ODE on the indicated interval.

$$\phi(t) = 3e^{2t}, \quad I = (-\infty, \infty), \quad \frac{d^2y}{dt^2} - \frac{dy}{dt} - 2y = 0$$

ϕ has derivatives of all orders,

Set $y = 3e^{2t}$, then $\frac{dy}{dt} = 6e^{2t}$ and $\frac{d^2y}{dt^2} = 12e^{2t}$

$$\frac{d^2y}{dt^2} - \frac{dy}{dt} - 2y \stackrel{?}{=} 0$$

$$12e^{2t} - 6e^{2t} - 2(3e^{2t}) =$$

$$e^{2t}(12 - 6 - 6) = 0 \quad \text{yes!}$$

Verify that the relation(left) defines and implicit solution of the differential equation (right).

$$y^2 - 2x^2 y = 1, \qquad \frac{dy}{dx} = \frac{2xy}{y - x^2}$$

We'll use implicit diff. to show that if y solves the relation, it also solves the ODE.

$$2y\frac{dy}{dx} - 4xy - 2x^2\frac{dy}{dx} = 0$$

$$(2y - 2x^2)\frac{dy}{dx} = 4xy$$

$$2(y - x^2)\frac{dy}{dx} = 4xy \Rightarrow \frac{dy}{dx} = \frac{2xy}{y - x^2}$$

as expected.

It may not be possible to clearly identify the domain of definition of an implicit solution.

Function vs Solution

The interval of defintion has to be an interval.

Consider $y' = -y^2$. Clearly $y = \frac{1}{x}$ solves the DE. The interval of defintion can be $(-\infty, 0)$, or $(0, \infty)$—or any interval that doesn't contain the origin. But it can't be $(-\infty, 0) \cup (0, \infty)$ because this isn't an interval!

Often, we'll take I to be the largest, or one of the largest, possible intervasl. It may depend on other information.

Solution curves won't be disconnected pieces

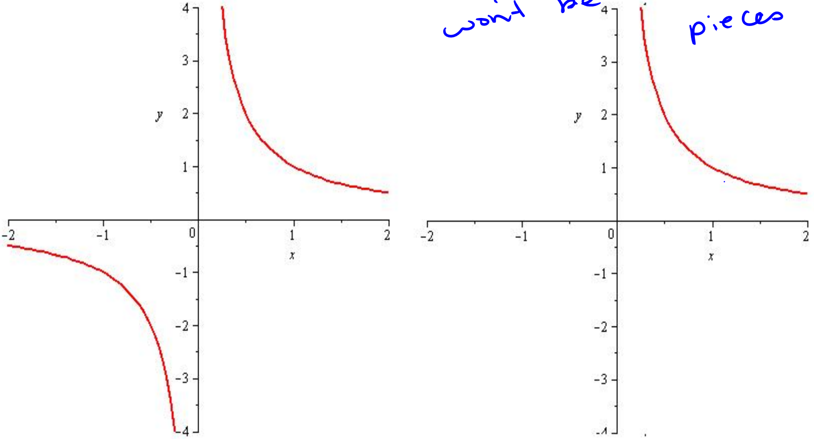

Figure: Left: Plot of $f(x) = \frac{1}{x}$ as a **function**. Right: Plot of $f(x) = \frac{1}{x}$ as a possible **solution** of an ODE.