August 15 Math 2306 sec. 57 Fall 2017

Section 1: Concepts and Terminology

Suppose $y = \phi(x)$ is a differentiable function. We know that $dy/dx = \phi'(x)$ is another (related) function.

For example, if $y = \cos(2x)$, then y is differentiable on $(-\infty, \infty)$. In fact,

$$\frac{dy}{dx} = -2\sin(2x).$$

Even dy/dx is differentiable with $d^2y/dx^2 = -4\cos(2x)$.

Suppose $y = \cos(2x)$

Note that $$\frac{d^2y}{dx^2} + 4y = 0.$$

We know that $\frac{d^2y}{dx^2} = -4\cos(2x)$

and $4y = 4\cos(2x)$

so $\frac{d^2y}{dx^2} + 4y = -4\cos(2x) + 4\cos(2x) = 0$

as expected

A *differential* equation

The equation

$$\frac{d^2y}{dx^2} + 4y = 0.$$

is an example of a **differential equation.**

Questions: If we only started with the equation, how could we determine that $\cos(2x)$ satisfies it? Also, is $\cos(2x)$ the only possible function that y could be?

We'll be able to answer these types of questions after this course.

Definition

A **Differential Equation** is an equation containing the derivative(s) of one or more dependent variables, with respect to one or more indendent variables.

Solving a differential equation refers to determining the dependent variable(s)—as function(s).

Independent Variable: will appear as one that derivatives are taken with respect to.

Dependent Variable: will appear as one that derivatives are taken of.

these are our functions

Independent and Dependent Variables

Often, the derivatives indicate which variable is which:

$$\frac{dy}{dx} \qquad \frac{du \leftarrow \text{dep.}}{dt_{\nwarrow \text{ind.}}} \qquad \frac{dx \leftarrow \text{dep.}}{dr_{\nwarrow \text{ind.}}}$$

the derivative of y with respect to x

y is dependent

x is independent

Classifications

Type: An **ordinary differential equation (ODE)** has exactly one independent variable[1]. For example

$$\frac{dy}{dx} - y^2 = 3x, \quad \text{or} \quad \frac{dy}{dt} + 2\frac{dx}{dt} = t, \quad \text{or} \quad y'' + 4y = 0$$

A **partial differential equation (PDE)** has two or more independent variables. For example

$$\frac{\partial y}{\partial t} = \frac{\partial^2 y}{\partial x^2}, \quad \text{or} \quad \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0$$

[1] These are the subject of this course.

Classifications

Order: The order of a differential equation is the same as the highest order derivative appearing anywhere in the equation.

$$\frac{dy}{dx} - y^2 = 3x$$ 1st order

$$y''' + (y')^4 = x^3$$ 3rd order

$$\frac{\partial y}{\partial t} = \frac{\partial^2 y}{\partial x^2}$$ 2nd order

Notations and Symbols

We'll use standard derivative notations:

$$\text{Leibniz:} \quad \frac{dy}{dx}, \quad \frac{d^2y}{dx^2}, \quad \ldots \quad \frac{d^ny}{dx^n}, \quad \text{or}$$

$$\text{Prime \& superscripts:} \quad y', \quad y'', \quad \ldots \quad y^{(n)}.$$

Newton's **dot notation** may be used if the independent variable is time. For example if s is a position function, then

$$\text{velocity is } \frac{ds}{dt} = \dot{s}, \quad \text{and acceleration is } \frac{d^2s}{dt^2} = \ddot{s}$$

Notations and Symbols

An n^{th} order ODE, with independent variable x and dependent variable y can always be expressed as an equation

$$F(x, y, y', \ldots, y^{(n)}) = 0$$

where F is some real valued function of $n + 2$ variables.

Normal Form: If it is possible to isolate the highest derivative term, then we can write a **normal form** of the equation

$$\frac{d^n y}{dx^n} = f(x, y, y', \ldots, y^{(n-1)}).$$

$$\frac{d^2y}{dx^2} + 4y = 0$$

This is of the form $F(x, y, y', y'') = 0$

where $F(x, y, y', y'') = y'' + 4y$

In normal form, this is

$$\frac{d^2y}{dx^2} = -4y \qquad \text{so } f(x, y, y') = -4y$$

Notations and Symbols

If $n = 1$ or $n = 2$, an equation in normal form would look like

$$\frac{dy}{dx} = f(x, y) \quad \text{or} \quad \frac{d^2y}{dx^2} = f(x, y, y').$$

Differential Form: A first order equation may appear in the form

$$\underbrace{M(x, y)\,dx + N(x, y)\,dy}_{\text{differential form}} = 0$$

$$M(x,y)\,dx + N(x,y)\,dy = 0$$

Differential forms may be written in normal form in a couple of ways.

If $N(x,y) \neq 0$, then $N(x,y)\,dy = -M(x,y)\,dx$

$$\frac{dy}{dx} = \frac{-M(x,y)}{N(x,y)}$$

Similarly if $M(x,y) \neq 0$

$$\frac{dx}{dy} = \frac{-N(x,y)}{M(x,y)}$$

Classifications

Linearity: An n^{th} order differential equation is said to be **linear** if it can be written in the form

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x).$$

Note that each of the coefficients $a_0, \ldots, a_n$ and the right hand side g may depend on the independent variable but not on the dependent variable or any of its derivatives.

Key: y and its derivatives can only appear to the 1st power. Not inside other functions or multiplied by each other.

Examples (Linear -vs- Nonlinear)

Both linear

$$y'' + 4y = 0$$

$a_2(x) = 1$

$a_1(x) = 0$ $\quad g(x) = 0$

$a_0(x) = 4$

$$t^2\frac{d^2x}{dt^2} + 2t\frac{dx}{dt} - x = e^t$$

$a_2(t) = t^2$ $\quad g(t) = e^t$

$a_1(t) = 2t$

$a_0(t) = -1$

both nonlinear

$$\frac{d^3y}{dx^3} + \left(\frac{dy}{dx}\right)^4 = x^3$$

a nonlinear term

$$u'' + u' = \underbrace{\cos u}_{\text{nonlinear term}}$$

dependent u is inside the cosine function.

Identify the independent and dependent variables. Determine the order of the equation. State whether it is linear or nonlinear.

(a) $y'' + 2ty' = \cos t + y - y'''$

3^{rd} order

dependent var. y

Independent var t

$$y''' + y'' + 2ty' - y = \cos t$$

It is linear

(b) $\ddot{\theta}+\dfrac{g}{\ell}\sin\theta = 0$ $\quad g$ and ℓ are constant

2^{nd} order

dependent var $\quad\theta$

independent var $\quad t$ for time

This is nonlinear due to the

$\sin\theta$ term.

Solution of $F(x, y, y', \dots, y^{(n)}) = 0$ (*)

Definition: A function ϕ defined on an interval[2] I and possessing at least n continuous derivatives on I is a **solution** of (*) on I if upon substitution (i.e. setting $y = \phi(x)$) the equation reduces to an identity.

Definition: An **implicit solution** of (*) is a relation $G(x, y) = 0$ provided there exists at least one function $y = \phi$ that satisfies both the differential equation (*) and this relation.

[2] The interval is called the *domain of the solution* or the *interval of definition*.

Examples:

Verify that the given function is an solution of the ODE on the indicated interval.

$$\phi(t) = 3e^{2t}, \quad I = (-\infty, \infty)\,, \quad \frac{d^2y}{dt^2} - \frac{dy}{dt} - 2y = 0$$

ϕ has derivatives of all orders on I.

Set $y = 3e^{2t}$, then $y' = 6e^{2t}$ and $y'' = 12e^{2t}$

$$\frac{d^2y}{dt^2} - \frac{dy}{dt} - 2y \stackrel{?}{=} 0$$

$$12e^{2t} - 6e^{2t} - 2(3e^{2t}) =$$

$$(12 - 6 - 6)\, e^{2t} =$$

yes it's a solution

$$0 = 0$$

Verify that the relation(left) defines and implicit solution of the differential equation (right).

$$y^2 - 2x^2y = 1, \qquad \frac{dy}{dx} = \frac{2xy}{y - x^2}$$

We'll use implicit differentiation to show that if y solves the relation, it solves the ODE.

$$2y\frac{dy}{dx} - 4xy - 2x^2\frac{dy}{dx} = 0$$

$$2(y-x^2)\frac{dy}{dx} = 4xy \implies \frac{dy}{dx} = \frac{2xy}{y-x^2}$$

this is the right ODE

It may not be possible to clearly identify the domain of definition of an implicit solution.

Function vs Solution

The interval of defintion has to be an interval.

Consider $y' = -y^2$. Clearly $y = \frac{1}{x}$ solves the DE. The interval of defintion can be $(-\infty, 0)$, or $(0, \infty)$—or any interval that doesn't contain the origin. But it can't be $(-\infty, 0) \cup (0, \infty)$ because this isn't an interval!

Often, we'll take I to be the largest, or one of the largest, possible intervasl. It may depend on other information.

Solution curves can't be in disconnected pieces

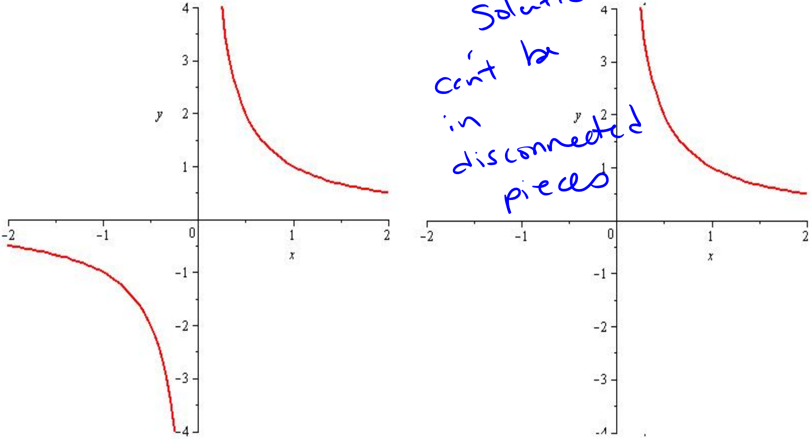

Figure: Left: Plot of $f(x) = \frac{1}{x}$ as a **function**. Right: Plot of $f(x) = \frac{1}{x}$ as a possible **solution** of an ODE.