August 15 Math 3260 sec. 57 Fall 2017

Section 1.1: Systems of Linear Equations

We begin with a linear (*algebraic*) equation in n variables x_1, x_2, ..., x_n for some positive integer n.

A **linear equation** can be written in the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b.$$

The numbers $a_1, \ldots, a_n$ are called the *coefficients*. These numbers and the right side b are real (or complex) constants that are **known**.

Examples of Equations that are or are not Linear

Both linear

$$2x_1 = 4x_2 - 3x_3 + 5 \quad \text{and} \quad 12 - \sqrt{3}(x + y) = 0$$

$2x_1 - 4x_2 + 3x_3 = 5$

$\sqrt{3}\,x + \sqrt{3}\,y = 12$

both nonlinear

$$x_1 + 3x_3 = \frac{1}{x_2} \quad \text{and} \quad xyz = \sqrt{w}$$

non linear term

non linear on both sides

A *Linear System* is a collection of linear equations in the same variables

$$\begin{array}{rcrcrcrcr} 2x_1 & + & x_2 & - & 3x_3 & + & x_4 & = & -3 \\ -x_1 & + & 3x_2 & + & 4x_3 & - & 2x_4 & = & 8 \end{array}$$

$$\begin{array}{rcrcrcr} x & + & 2y & + & 3z & = & 4 \\ 3x & & & + & 12z & = & 0 \\ 2x & + & 2y & - & 5z & = & -6 \end{array}$$

Some terms

- A **solution** is a list of numbers $(s_1, s_2, \dots, s_n)$ that reduce each equation in the system to a true statement upon substitution.
- A **solutions set** is the set of all possible solutions of a linear system.
- Two systems are called **equivalent** if they have the same solution set.

An Example

$$\begin{array}{rcrcr} 2x & - & y & = & -1 \\ -4x & + & 2y & = & 2 \end{array}$$

(a) Show that $(1, 3)$ is a solution.

Here $x=1$ and $y=3$ in the pair $(1,3)$.

$2(1) - 3 = 2-3 = -1$ $\quad -1=-1$ the 1^{st} equation is satisfied

$-4(1) + 2(3) = -4+6 = 2$ $\quad 2=2$ the 2^{nd} eqn is satisfied

Both equations hold, so $(1,3)$ is a solution.

An Example Continued

$$\begin{array}{rcrcr} 2x & - & y & = & -1 \\ -4x & + & 2y & = & 2 \end{array}$$

(b) Note that $\{(x, y) | y = 2x + 1\}$ is the solution set.

From $2x - y = -1 \Rightarrow 2x = -1 + y \Rightarrow 2x + 1 = y$

From $-4x + 2y = 2 \Rightarrow 2y = 2 + 4x$

$2y = 2(1 + 2x) \Rightarrow y = 1 + 2x$

The Geometry of 2 Equations with 2 Variables

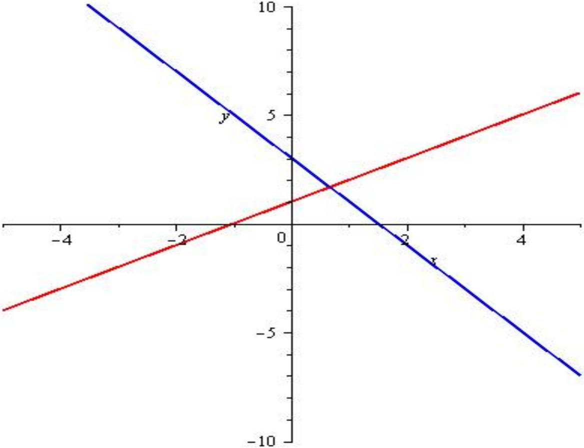

Figure: The system $x - y = -1$ and $2x + y = 3$ with solution set $\{(2/3, 5/2)\}$. These equations represent lines that intersect at one point.

The Geometry of 2 Equations with 2 Variables

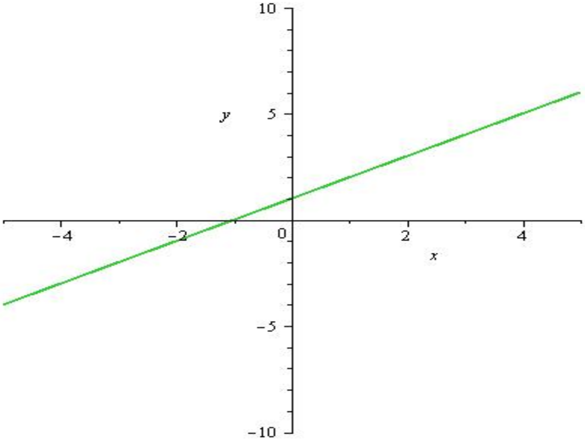

Figure: The system $x - y = -1$ and $2x - 2y = -2$ with solution set $\{(x, y) | y = x + 1\}$. Both equations represent the same line which share all common points as solutions.

The Geometry of 2 Equations with 2 Variables

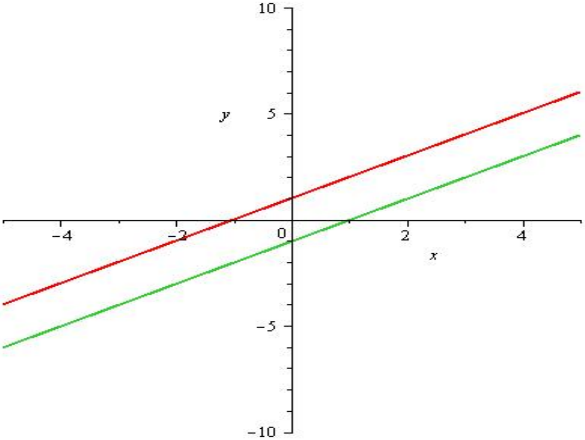

Figure: The system $x - y = -1$ and $2x - 2y = 2$ with solution set $\emptyset$. These equations represent parallel lines having no common points.

Theorem

A linear system of equations has exactly one of the following:

i. No solution, or
ii. Exactly one solution, or
iii. Infinitely many solutions.

Terms: A system is
consistent if it has at least one solution (cases ii and iii), and
inconsistent if is has no solutions (case i).

Two critical questions about any linear system are: (1) Does it have a solution? (existence), and (2) If it has a solution, is there only one? (uniqueness)

Matrices

Definition: A matrix is a rectangular array of numbers. It's **size** (a.k.a. dimension/order) is $m \times n$ (read "m by n") where m is the number of rows and n is the number of columns the matrix has.

Examples:

$$\begin{bmatrix} 2 & 0 & -1 & 3 \\ 1 & 1 & 13 & -4 \\ 12 & -3 & 2 & -2 \end{bmatrix}, \qquad \begin{bmatrix} 2 & 0 \\ 4 & 4 \\ 3 & -5 \end{bmatrix}$$

3×4 $\qquad$ 3×2

Linear System: Coefficient Matrix

Given any linear system of equations, we can associate two matrices with the system. These are the **coefficient** matrix and the **augmented** matrix[1].

Example:
$$\begin{array}{rcrcrcr} x_1 & + & 2x_2 & - & x_3 & = & -4 \\ 2x_1 & & & + & x_3 & = & 7 \\ x_1 & + & x_2 & + & x_3 & = & 6 \end{array}$$

The coefficient matrix holds the coefficients of the left side.

$$\begin{bmatrix} 1 & 2 & -1 \\ 2 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

m = # equations

n = # variables

[1] Note that like variables should be lined up vertically!

Linear System: Augmented Matrix

Given any linear system of equations, we can associate two matrices with the system. These are the **coefficient** matrix and the **augmented** matrix.

$$\text{Example:}\quad \begin{array}{rcrcrcr} x_1 & + & 2x_2 & - & x_3 & = & -4 \\ 2x_1 & & & + & x_3 & = & 7 \\ x_1 & + & x_2 & + & x_3 & = & 6 \end{array}$$

Augmented is the coefficient matrix with one extra column for the right side.

$$\begin{bmatrix} 1 & 2 & -1 & -4 \\ 2 & 0 & 1 & 7 \\ 1 & 1 & 1 & 6 \end{bmatrix}$$

m = # equations

n = # variables + 1

Legitimate Operations for Solving a System

We can perform three basic operation **without** changing the solution set of a system. These are

- swap the order of any two equations **(swap)**,
- multiply an equation by any nonzero constant **(scale)**, and
- replace an equation with the sum of itself and a nonzero multiple of any other equation **(replace)**.

Some Operation Notation

Notation

- Swap equations i and j:
$$Ei \leftrightarrow Ej$$
- Scale equation i by k:
$$kEi \rightarrow Ei$$
- Replace equation j with the sum of itself and k times equation i:
$$kEi + Ej \rightarrow Ej$$

Solve the following system of equations by *elimination*. Keep tabs on the augmented matrix at each step.

$$\begin{array}{rcrcrcr} x_1 & + & 2x_2 & - & x_3 & = & -4 \\ 2x_1 & & & + & x_3 & = & 7 \\ x_1 & + & x_2 & + & x_3 & = & 6 \end{array}$$

$$\begin{bmatrix} 1 & 2 & -1 & -4 \\ 2 & 0 & 1 & 7 \\ 1 & 1 & 1 & 6 \end{bmatrix}$$

$-2E1 + E2 \to E2$

$-E1 + E3 \to E3$

$$\begin{array}{rcrcrcr} x_1 & + & 2x_1 & - & x_3 & = & -4 \\ & & -4x_2 & + & 3x_3 & = & 15 \\ & & -x_2 & + & 2x_3 & = & 10 \end{array}$$

$$\begin{bmatrix} 1 & 2 & -1 & -4 \\ 0 & -4 & 3 & 15 \\ 0 & -1 & 2 & 10 \end{bmatrix}$$

$E2 \leftrightarrow E3$

then $-E2 \to E2$

$$\begin{aligned} x_1 + 2x_2 - x_3 &= -4 \\ x_2 - 2x_3 &= -10 \\ -4x_2 + 3x_3 &= 15 \end{aligned}$$

$$\begin{bmatrix} 1 & 2 & -1 & -4 \\ 0 & 1 & -2 & -10 \\ 0 & -4 & 3 & 15 \end{bmatrix}$$

$4E2 + E3 \to E3$

$$\begin{aligned} x_1 + 2x_2 - x_3 &= -4 \\ x_2 - 2x_3 &= -10 \\ -5x_3 &= -25 \end{aligned}$$

$$\begin{bmatrix} 1 & 2 & -1 & -4 \\ 0 & 1 & -2 & -10 \\ 0 & 0 & -5 & -25 \end{bmatrix}$$

$-\frac{1}{5}E3 \to E3$

$$\begin{aligned} x_1 + 2x_2 - x_3 &= -4 \\ x_2 - 2x_3 &= -10 \\ x_3 &= 5 \end{aligned}$$

$$\begin{bmatrix} 1 & 2 & -1 & -4 \\ 0 & 1 & -2 & -10 \\ 0 & 0 & 1 & 5 \end{bmatrix}$$

From E3, $x_3 = 5$

From E2, $x_2 = -10 + 2x_3 = -10 + 2(5) = 0$

From E1, $x_1 = -4 - 2x_2 + x_3 = -4 - 0 + 5 = 1$

we can express the solution as the triple $(1, 0, 5)$.

Elementary Row Operations

If any sequence of the following operations are performed on a matrix, the resulting matrix is **row equivalent**.

i Replace a row with the sum of itself and a multiple of another row (**replacement**).

ii Interchange any two rows (**row swap**).

iii Multiply a row by any nonzero constant (**scaling**).

Theorem: If the augmented matrices of two linear systems are row equivalent, then the systems have the same solution set. (i.e. The systems are equivalent!)

A key here is *structure*!

Consider the following augmented matrix. Determine if the associated system is consistent or inconsistent. If it is consistent, determine the solution set.

(a) $\left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -2 \end{array}\right]$

$x_1 = 3$

$x_2 = 1$

$x_3 = -2$

It is consistent with solution $x_1 = 3$, $x_2 = 1$, $x_3 = -2$

(b) $\begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 1 & -1 & 4 \\ 0 & 0 & 0 & 3 \end{bmatrix}$

$$\begin{aligned} x_1 + 2x_2 \qquad &= 3 \\ x_2 - x_3 &= 4 \\ 0 &= 3 \end{aligned}$$

The third equation is always false.

The system is inconsistent.

(c) $\begin{bmatrix} 1 & 1 & -1 & 0 \\ 0 & 1 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$$x_1 + x_2 - x_3 = 0$$
$$x_2 + x_3 = 4$$
$$0 = 0 \leftarrow \text{always true}$$

The system is consistent

$x_1 = -x_2 + x_3$ $\quad$ $x_2 = 4 - x_3$

so $\quad x_1 = -(4 - x_3) + x_3 = -4 + 2x_3$

The solution set can be written as

$$x_1 = -4 + 2x_3$$
$$x_2 = 4 - x_3$$
x_3 is free $\quad$ this can be written as $x_3 \in \mathbb{R}$

In this example, x_3 is called free

and x_1 and x_2 are called

basic.