August 17 Math 2306 sec. 53 Fall 2018

Section 1: Concepts and Terminology

We consider an n^{th} order ODE $F(x, y, y', \dots, y^{(n)}) = 0$ (*)

Definition: A function ϕ defined on an interval[1] I and possessing at least n continuous derivatives on I is a **solution** of (*) on I if upon substitution (i.e. setting $y = \phi(x)$) the equation reduces to an identity.

Definition: An **implicit solution** of (*) is a relation $G(x, y) = 0$ provided there exists at least one function $y = \phi$ that satisfies both the differential equation (*) and this relation.

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[1] The interval is called the *domain of the solution* or the *interval of definition*.

Function vs Solution

The interval of defintion has to be an interval.

Example: Consider the ODE $y' = -y^2$. A solution is $y = \frac{1}{x}$.

The interval of defintion can be

$$(-\infty, 0) \quad \text{or} \quad (0, \infty)$$

or any interval that doesn't contain the origin.

But it CAN't be $(-\infty, 0) \cup (0, \infty)$ because this isn't an interval!

Depending on other available information, we often assume I is the largest (or one of the largest) possible intervals.

Function vs Solution

The graph of an ODE solution[2] will not have disconnected pieces.

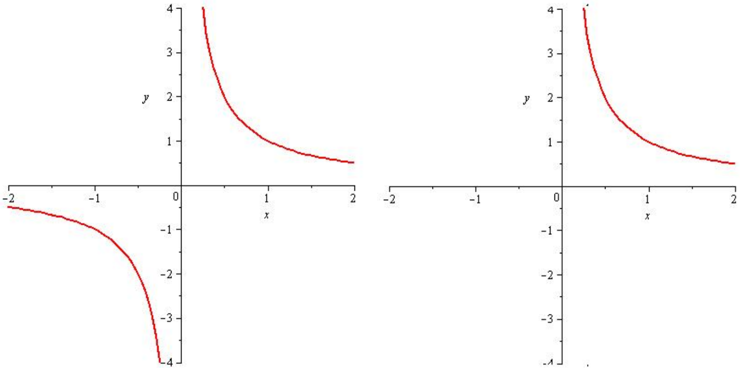

Figure: Left: Plot of $f(x) = \frac{1}{x}$ as a **function**. Right: Plot of $f(x) = \frac{1}{x}$ as a possible **solution** of an ODE.

[2]This is known as a *classical solution*.

Unspecified Constants in a Function

Show that for any choice of constants c_1 and c_2, $y = c_1 x + \frac{c_2}{x}$ is a solution of the differential equation

$$x^2 y'' + xy' - y = 0$$

we'll substitute:

$$y = c_1 x + c_2 x^{-1}$$

$$y' = c_1 - c_2 x^{-2}$$

$$y'' = 0 + 2c_2 x^{-3}$$

$$x^2 y'' + xy' - y = x^2(2c_2 x^{-3}) + x(c_1 - c_2 x^{-2}) - (c_1 x + c_2 x^{-1})$$

$$= 2c_2 x^{-1} + c_1 x - c_2 x^{-1} - c_1 x - c_2 x^{-1}$$

$$= x^{-1}\left(2c_2 - c_2 - c_2\right) + x\left(c_1 - c_1\right)$$

$$= 0\cdot x^{-1} + 0x = 0 \quad \text{as expected!}$$

So for any numbers c_1, c_2

$y = c_1 x + \dfrac{c_2}{x}$ solves the ODE.

Some Terms

- A **parameter** is an unspecified constant such as c_1 and c_2 in the last example.
- A **family of solutions** is a collection of solution functions that only differ by a parameter.
- An ***n*-parameter family of solutions** is one containing n parameters (e.g. $c_1x + \frac{c_2}{x}$ is a 2 parameter family).
- A **particular solution** is one with no arbitrary constants in it.
- The **trivial solution** is the simple constant function $y = 0$.
- An **integral curve** is the graph of one solution (perhaps from a family).

Section 2: Initial Value Problems

IVP

An initial value problem consists of an ODE with a certain type of additional conditions.

Solve the equation [3]

$$\frac{d^n y}{dx^n} = f(x, y, y', \ldots, y^{(n-1)}) \tag{1}$$

subject to the *initial conditions*

$$y(x_0) = y_0, \quad y'(x_0) = y_1, \quad \ldots, y^{(n-1)}(x_0) = y_{n-1}. \tag{2}$$

The problem (1)–(2) is called an *initial value problem* (IVP).

[3] on some interval I containing x_0.

IVPs

First order case:

$$\frac{dy}{dx} = f(x,y), \quad y(x_0) = y_0$$

1st order ODE

one condition

(x_0, y_0) is a point on the graph of y

Second order case:

2nd order ODE

two conditions

$$\frac{d^2y}{dx^2} = f(x,y,y'), \quad y(x_0) = y_0, \quad y'(x_0) = y_1$$

If y is the position of a particle @ time x

y'' is the acceleration

y_0 - initial position, y_1 - initial velocity

Example

Given that $y = c_1 x + \frac{c_2}{x}$ is a 2-parameter family of solutions of $x^2 y'' + xy' - y = 0$, solve the IVP

$$x^2 y'' + xy' - y = 0, \quad y(1) = 1, \quad y'(1) = 3$$

The solutions to the ODE look like

$y = c_1 x + \dfrac{c_2}{x}$. We need to figure out c_1, c_2 values that satisfy the initial conditions (IC).

Impose $y(1) = 1$ and $y'(1) = 3$

$$y(1) = c_1(1) + \frac{c_2}{1} = 1 \Rightarrow c_1 + c_2 = 1$$

(from before $y' = c_1 - c_2 x^{-2}$)

$$y'(1) = c_1 - c_2(1)^{-2} = 3 \Rightarrow c_1 - c_2 = 3$$

we solve this system

add eqns: $2c_1 = 4$

$c_1 = 2$

from equation 1: $2 + c_2 = 1 \Rightarrow c_2 = 1 - 2 = -1$

The solution to the IVP is

$$y = 2x - \frac{1}{x}$$