August 17 Math 2306 sec. 56 Fall 2017

Section 1: Concepts and Terminology

Recall that a **Differential Equation** is an equation containing the derivative(s) of one or more dependent variables, with respect to one or more indendent variables.

$$\text{For the general } n^{th} \text{ order ODE} \quad F(x, y, y', \dots, y^{(n)}) = 0 \quad (*)$$

Definition: A function ϕ defined on an interval I and possessing at least n continuous derivatives on I is a **solution** of (*) on I if upon substitution (i.e. setting $y = \phi(x)$) the equation reduces to an identity.

Definition: An **implicit solution** of (*) is a relation $G(x, y) = 0$ provided there exists at least one function $y = \phi$ that satisfies both the differential equation (*) and this relation.

Example

Show that for any choice of constants c_1 and c_2, $y = c_1 x + \frac{c_2}{x}$ is a solution of the differential equation

$$x^2 y'' + xy' - y = 0$$

$y = c_1 x + c_2 x^{-1}$

$y' = c_1 - c_2 x^{-2}$

$y'' = 2 c_2 x^{-3}$

$x^2 y'' + x y' - y =$

$x^2 (2 c_2 x^{-3}) + x (c_1 - c_2 x^{-2}) - (c_1 x + c_2 x^{-1}) =$

$$2c_2x^{-1} + c_1x - c_2x^{-1} - c_1x - c_2x^{-1} =$$

$$c_2x^{-1}(2-1-1) + c_1x(1-1) =$$

$$c_2x^{-1}(0) + c_1x(0) = 0$$ as required.

So $y = c_1x + \frac{c_2}{x}$ solves the ODE for any choice of c_1 and c_2.

Some Terms

- A **parameter** is an unspecified constant such as c_1 and c_2 in the last example.

- A **family of solutions** is a collection of solution functions that only differ by a parameter.
- An ***n*-parameter family of solutions** is one containing n parameters (e.g. $c_1x + \frac{c_2}{x}$ is a 2 parameter family).

- A **particular solution** is one with no arbitrary constants in it.

- The **trivial solution** is the simple constant function $y = 0$.

- An **integral curve** is the graph of one solution (perhaps from a family).

Section 2: Initial Value Problems

IVP

An initial value problem consists of an ODE with additional conditions.

Solve the equation [1]

$$\frac{d^n y}{dx^n} = f(x, y, y', \ldots, y^{(n-1)}) \tag{1}$$

subject to the *initial conditions*

$$y(x_0) = y_0, \quad y'(x_0) = y_1, \quad \ldots, y^{(n-1)}(x_0) = y_{n-1}. \tag{2}$$

The problem (1)–(2) is called an *initial value problem* (IVP).

[1] on some interval I containing x_0.

First order case:

$$\frac{dy}{dx} = f(x, y), \quad y(x_0) = y_0$$

1st order ODE + one initial condition

Second order case: 2nd order ODE + 2 initial conditions

$$\frac{d^2y}{dx^2} = f(x, y, y'), \quad y(x_0) = y_0, \quad y'(x_0) = y_1$$

If y is the position of a moving particle

The ODE gives acceleration

$y(x_0)$ = initial position, $y'(x_0)$ = initial velocity

Example

Given that $y = c_1 x + \frac{c_2}{x}$ is a 2-parameter family of solutions of $x^2y'' + xy' - y = 0$, solve the IVP

$$x^2y'' + xy' - y = 0, \quad y(1) = 1, \quad y'(1) = 3$$

We already know that $y = c_1 x + \frac{c_2}{x}$, we need to impose the conditions $y(1)=1$ and $y'(1)=3$.

$$y = c_1 x + \frac{c_2}{x} \qquad y(1) = c_1 1 + \frac{c_2}{1} = c_1 + c_2 = 1$$

$$y' = c_1 - \frac{c_2}{x^2} \qquad y'(1) = c_1 - \frac{c_2}{1^2} = \underbrace{c_1 - c_2 = 3}_{\text{2 eqns for } c_1 \text{ and } c_2}$$

$$c_1 + c_2 = 1$$

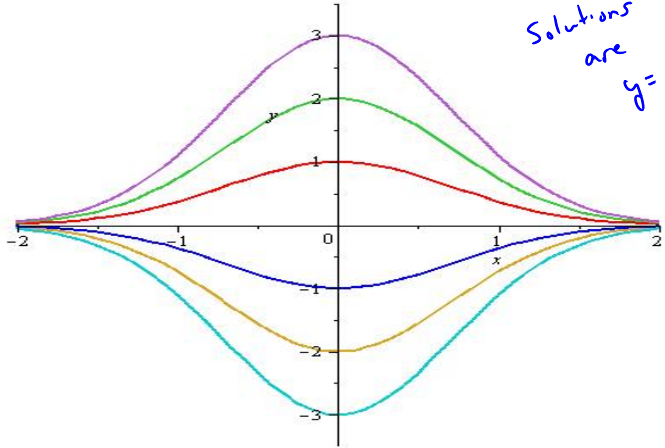
add $$c_1 - c_2 = 3$$

$$2c_1 = 4 \implies c_1 = 2$$

$$c_2 = 1 - c_1 = 1 - 2 = -1$$

$$\boxed{\text{The solution to the IVP is} \quad y = 2x - \frac{1}{x}.}$$

Graphical Interpretation



Solutions are $y = Ce^{-x^2}$

Figure: Each curve solves $y' + 2xy = 0$, $y(0) = y_0$. Each colored curve corresponds to a different value of y_0

Example

Part 1

Show that for any constant c the relation $x^2 + y^2 = c$ is an implicit solution of the ODE

$$\frac{dy}{dx} = -\frac{x}{y}$$

Using implicit Diff: $2x + 2y\frac{dy}{dx} = 0$

$$2y\frac{dy}{dx} = -2x \Rightarrow \frac{dy}{dx} = \frac{-2x}{2y}$$

$$\Rightarrow \frac{dy}{dx} = \frac{-x}{y}$$ as expected

Example

Part 2

Use the preceding results to find an explicit solution of the IVP

$$\frac{dy}{dx} = -\frac{x}{y}, \quad y(0) = -2$$

we know the solutions are defined by

$$x^2 + y^2 = C$$

using the condition $y(0) = -2$

$$0^2 + (-2)^2 = C \implies C = 4$$

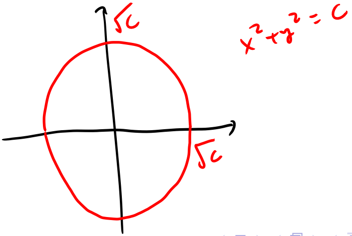
$x^2 + y^2 = 4$ solves the IVP implicitly.

This defines 2 functions

$$y = \sqrt{4 - x^2} \quad \text{or} \quad y = -\sqrt{4 - x^2}$$

Since $y(0) = -2$, the explicit solution is

$$y = -\sqrt{4 - x^2}$$



Example

$x = c_1 \cos(2t) + c_2 \sin(2t)$ is a 2-parameter family of solutions of the ODE $x'' + 4x = 0$. Find a solution of the IVP

$$x'' + 4x = 0, \quad x\left(\frac{\pi}{2}\right) = -1, \quad x'\left(\frac{\pi}{2}\right) = 4$$

$x = c_1 \cos(2t) + c_2 \sin(2t)$

$x' = -2c_1 \sin(2t) + 2c_2 \cos(2t)$

$x\left(\frac{\pi}{2}\right) = c_1 \cos\left(2 \cdot \frac{\pi}{2}\right) + c_2 \sin\left(2 \cdot \frac{\pi}{2}\right) = -c_1 = -1$

$c_1 = 1$

$$x'\left(\frac{\pi}{2}\right) = -2c_1 \sin(2\cdot\pi/2) + 2c_2\cos\left(2\cdot\frac{\pi}{2}\right) = -2c_2 = 4$$

$$c_2 = -2$$

The solution to the IVP is

$$\boxed{x = \cos(2t) - 2\sin(2t).}$$

Existence and Uniqueness

Two important questions we can always pose (and sometimes answer) are

(1) Does an IVP have a solution? (existence) and

(2) If it does, is there just one? (uniqueness)

Hopefully it's obvious that we can't solve $\left(\frac{dy}{dx}\right)^2 + 1 = -y^2$.

always positive

never positive

Uniqueness

Consider the IVP

$$\frac{dy}{dx} = x\sqrt{y} \quad y(0) = 0$$

Verify that $y = \frac{x^4}{16}$ is a solution of the IVP. And find a second solution of the IVP by clever guessing.

Initial cond. $y(0)=0$ $\qquad y(0) = \frac{0^4}{16} = 0$ so it solves the IC

How about the ODE:

$$\frac{dy}{dx} = \frac{4x^3}{16} = \frac{x^3}{4}, \qquad \sqrt{y} = \sqrt{\frac{x^4}{16}} = \frac{x^2}{4}$$

so $\frac{dy}{dx} = \frac{x^3}{4} = x\left(\frac{x^2}{4}\right) = x\sqrt{y}$ as expected

Indeed $y = \frac{x^4}{16}$ solves the IVP

$$\frac{dy}{dx} = x\sqrt{y}, \quad y(0) = 0$$

Maybe there's a constant solution $y = c$.

$y = 0$ works

Note $y(0) = 0$

If $y = 0$, $y' = 0$ so $y' = 0 = x\sqrt{0} = x\sqrt{y}$

Section 3: Separation of Variables

The simplest type of equation we could encounter would be of the form

$$\frac{dy}{dx} = g(x).$$

For example, solve the ODE

$$\frac{dy}{dx} = 4e^{2x} + 1. \quad \Rightarrow \quad y = \int (4e^{2x} + 1)\, dx$$

$$= \frac{4}{2} e^{2x} + x + C$$

$$\boxed{y = 2e^{2x} + x + C}$$

$$* \int e^{ax}\, dx = \frac{1}{a} e^{ax} + C$$

for $a \neq 0$

Separable Equations

Definition: The first order equation $y' = f(x, y)$ is said to be **separable** if the right side has the form

$$f(x, y) = g(x)h(y).$$

That is, a separable equation is one that has the form

$$\frac{dy}{dx} = g(x)h(y).$$

Determine which (if any) of the following are separable.

(a) $\dfrac{dy}{dx} = x^3 y$

yes separable
with $g(x) = x^3$ and $h(y) = y$

(b) $\dfrac{dy}{dx} = 2x + y$

This is not separable it
can't be written as
$g(x)\,h(y)$