Section 1: Concepts and Terminology

Recall that a **Differential Equation** is an equation containing the derivative(s) of one or more dependent variables, with respect to one or more independent variables.

For the general $n^{th}$ order ODE

$$F(x, y, y', \ldots, y^{(n)}) = 0 \quad (*)$$

**Definition:** A function $\phi$ defined on an interval $I$ and possessing at least $n$ continuous derivatives on $I$ is a **solution** of $(*)$ on $I$ if upon substitution (i.e. setting $y = \phi(x)$) the equation reduces to an identity.

**Definition:** An **implicit solution** of $(*)$ is a relation $G(x, y) = 0$ provided there exists at least one function $y = \phi$ that satisfies both the differential equation $(*)$ and this relation.
Example

Show that for any choice of constants $c_1$ and $c_2$, $y = c_1 x + \frac{c_2}{x}$ is a solution of the differential equation

$$x^2 y'' + xy' - y = 0$$

\[
y = c_1 x + c_2 x^{-1}
\]
\[
y' = c_1 - c_2 x^{-2}
\]
\[
y'' = 2 c_2 x^{-3}
\]

$$x^2 y'' + xy' - y =$$

$$x^2 \left(2 c_2 x^{-3}\right) + x \left(c_1 - c_2 x^{-2}\right) - \left(c_1 x + c_2 x^{-1}\right) =$$
2 c_2 x^{-1} + c_1 x - c_2 x^{-1} - c_1 x - c_2 x^{-1} =

\begin{align*}
c_2 x^{-1} (2 - 1 - 1) + c_1 x (1 - 1) &= \\
c_2 x^{-1} (0) + c_1 x (0) &= 0 \quad \text{as required.}
\end{align*}

So, \( y = c_1 x + \frac{c_2}{x} \) solves the ODE for any choice of \( c_1 \) and \( c_2 \).
Some Terms

- **Parameter** is an unspecified constant such as \( c_1 \) and \( c_2 \) in the last example.

- **Family of solutions** is a collection of solution functions that only differ by a parameter.

- An **\( n \)-parameter family of solutions** is one containing \( n \) parameters (e.g. \( c_1 x + \frac{c_2}{x} \) is a 2 parameter family).

- **Particular solution** is one with no arbitrary constants in it.

- The **trivial solution** is the simple constant function \( y = 0 \).

- An **integral curve** is the graph of one solution (perhaps from a family).
Section 2: Initial Value Problems

An initial value problem consists of an ODE with additional conditions.

Solve the equation

\[ \frac{d^n y}{dx^n} = f(x, y, y', \ldots, y^{(n-1)}) \]  \hspace{1cm} (1)

subject to the initial conditions

\[ y(x_0) = y_0, \quad y'(x_0) = y_1, \quad \ldots, \quad y^{(n-1)}(x_0) = y_{n-1}. \]  \hspace{1cm} (2)

The problem (1)–(2) is called an initial value problem (IVP).

\footnote{on some interval \( I \) containing \( x_0 \).}
First order case:
\[ \frac{dy}{dx} = f(x, y), \quad y(x_0) = y_0 \]

Second order case:
\[ \frac{d^2y}{dx^2} = f(x, y, y'), \quad y(x_0) = y_0, \quad y'(x_0) = y_1 \]

If \( y \) is the position of a moving particle
The ODE gives acceleration
\( y(x_0) = \) initial position, \( y'(x_0) = \) initial velocity
Example

Given that \( y = c_1 x + \frac{c_2}{x} \) is a 2-parameter family of solutions of 
\[ x^2 y'' + xy' - y = 0, \]
solve the IVP

\[ x^2 y'' + xy' - y = 0, \quad y(1) = 1, \quad y'(1) = 3 \]

We already know that \( y = c_1 x + \frac{c_2}{x} \), we need to impose the conditions \( y(1) = 1 \) and \( y'(1) = 3 \).

\[
\begin{align*}
  y &= c_1 x + \frac{c_2}{x} \\
  y' &= c_1 - \frac{c_2}{x^2}
\end{align*}
\]

\[
\begin{align*}
  y(1) &= c_1 1 + \frac{c_2}{1} = c_1 + c_2 = 1 \\
  y'(1) &= c_1 - \frac{c_2}{1^2} = c_1 - c_2 = 3
\end{align*}
\]

2 eqns for \( c_1 \) and \( c_2 \)
\[ c_1 + c_2 = 1 \]
\[ c_1 - c_2 = 3 \]

\[
\text{add}
\]
\[ 2c_1 = 4 \quad \Rightarrow \quad c_1 = 2 \]
\[ c_2 = 1 - c_1 = 1 - 2 = -1 \]

The solution to the IVP is
\[ y = 2x - \frac{1}{x} \]
Graphical Interpretation

Figure: Each curve solves \( y' + 2xy = 0, \ y(0) = y_0 \). Each colored curve corresponds to a different value of \( y_0 \)

Solutions are \( y = Ce^{-x^2} \)
Example

Part 1

Show that for any constant $c$ the relation $x^2 + y^2 = c$ is an implicit solution of the ODE

$$\frac{dy}{dx} = -\frac{x}{y}$$

Using implicit Diff:

$$2x + 2y \frac{dx}{dx} = 0$$

$$2y \frac{dy}{dx} = -2x \implies \frac{dy}{dx} = -\frac{2x}{2y}$$

$$\implies \frac{dy}{dx} = \frac{-x}{y} \quad \text{as expected}$$
Example

Part 2

Use the preceding results to find an explicit solution of the IVP

\[ \frac{dy}{dx} = -\frac{x}{y}, \quad y(0) = -2 \]

we know the solutions are defined by

\[ x^2 + y^2 = C \]

using the condition \( y(0) = -2 \)

\[ 0^2 + (-2)^2 = C \quad \implies \quad C = 4 \]

\[ x^2 + y^2 = 4 \] solves the IVP implicitly.
This defines 2 functions

\[ y = \sqrt{4 - x^2} \quad \text{or} \quad y = -\sqrt{4 - x^2} \]

Since \( y(0) = -2 \), the explicit solution is

\[ y = -\sqrt{4 - x^2} \]

\[ x^2 + y^2 = c \]
Example

$x = c_1 \cos(2t) + c_2 \sin(2t)$ is a 2-parameter family of solutions of the ODE $x'' + 4x = 0$. Find a solution of the IVP

$$x'' + 4x = 0, \quad x\left(\frac{\pi}{2}\right) = -1, \quad x'\left(\frac{\pi}{2}\right) = 4$$

\[X = c_1 \cos(2t) + c_2 \sin(2t)\]

\[X' = -2c_1 \sin(2t) + 2c_2 \cos(2t)\]

\[X\left(\frac{\pi}{2}\right) = c_1 \cos\left(2 \cdot \frac{\pi}{2}\right) + c_2 \sin\left(2 \cdot \frac{\pi}{2}\right) = -c_1 = -1\]

\[c_1 = 1\]
\[ x'(\frac{\pi}{2}) = -2c_1 \sin(2\frac{\pi}{2}) + 2c_2 \cos(2\frac{\pi}{2}) = -2c_2 = y \]

\[ c_2 = -2 \]

The solution to the IVP is

\[ x = \cos(2t) - 2 \sin(2t). \]
Existence and Uniqueness

Two important questions we can always pose (and sometimes answer) are

(1) Does an IVP have a solution? (existence) and
(2) If it does, is there just one? (uniqueness)

Hopefully it’s obvious that we can’t solve \( \left( \frac{dy}{dx} \right)^2 + 1 = -y^2 \).
Uniqueness
Consider the IVP
\[ \frac{dy}{dx} = x \sqrt{y} \quad y(0) = 0 \]

Verify that \( y = \frac{x^4}{16} \) is a solution of the IVP. And find a second solution of the IVP by clever guessing.

Initial cond. \( y(0) = 0 \) \( y(0) = \frac{0}{16} = 0 \) \( \rightarrow \) it solves the IC

How about the ODE:
\[ \frac{dy}{dx} = \frac{y x^3}{16} = \frac{x^3}{y} \]
\[ \int \sqrt y = \sqrt{\frac{x^4}{16}} = \frac{x^2}{4} \]

So \[ \frac{dy}{dx} = \frac{x^3}{y} = x \left( \frac{x^2}{4} \right) = x \sqrt{y} \] as expected
Indeed \( y = \frac{x^4}{16} \) solves the IVP

\[
\frac{dy}{dx} = x\sqrt{5}, \quad y(0) = 0
\]

Maybe there's a constant solution \( y = C \).

\( y = 0 \) works

Note \( y(0) = 0 \)

If \( y = 0 \), \( y' = 0 \) so \( y' = 0 = x\sqrt{0} = x\sqrt{5} \)
Section 3: Separation of Variables

The simplest type of equation we could encounter would be of the form

\[ \frac{dy}{dx} = g(x). \]

For example, solve the ODE

\[ \frac{dy}{dx} = 4e^{2x} + 1. \]

\[ \Rightarrow \quad y = \int (4e^{2x} + 1) \, dx \]

\[ = \frac{4}{2} e^{2x} + x + C \]

\[ = 2e^{2x} + x + C \]

\[ y = 2e^{2x} + x + C \] for \( a \neq 0 \)
Separable Equations

**Definition:** The first order equation \( y' = f(x, y) \) is said to be **separable** if the right side has the form

\[
 f(x, y) = g(x)h(y).
\]

That is, a separable equation is one that has the form

\[
 \frac{dy}{dx} = g(x)h(y).
\]
Determine which (if any) of the following are separable.

(a) \( \frac{dy}{dx} = x^3 y \)  
   \( \text{yes separable} \)  
   \( \text{with } g(x) = x^3 \)  
   \( \text{and } h(y) = y \)  

(b) \( \frac{dy}{dx} = 2x + y \)  
   \( \text{This is not separable it} \)  
   \( \text{can't be written as } g(x) \ h(y) \)