August 17 Math 3260 sec. 57 Fall 2017

Section 1.2: Row Reduction and Echelon Forms

Recall: If two matrices are row equivalent, then the linear systems for which they are the augmented matrices are equivalent.

E.g. these are row equivalent

$$\begin{bmatrix} 2 & 2 & 11 & 3 & 4 \\ 3 & 2 & 14 & 4 & 2 \\ 1 & 1 & 6 & 1 & 2 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 0 & 4 & -2 \\ 0 & 1 & 0 & 3 & 4 \\ 0 & 0 & 1 & -1 & 0 \end{bmatrix}$$

Echelon Forms

ref

Definition: A matrix is in **echelon form** (a.k.a. **row echelon form**) if the following properties hold

i Any row of all zeros are at the bottom.

ii The first nonzero number (called the *leading entry*) in a row is to the right of the first nonzero number in all rows above it.

iii All entries below a leading entry are zeros.

Is

$$\begin{bmatrix} 2 & 1 & 3 \\ 0 & -1 & 1 \\ 0 & 0 & 7 \end{bmatrix}$$

Is Not

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & -3 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

Reduced Echelon Form

rref

Definition: A matrix is in **reduced echelon form** (a.k.a. **reduced row echelon form**) if it is in echelon form and the following additional properties hold

iv The leading entry of each row is 1 (called a *leading* 1), and

v each leading 1 is the only nonzero entry in its column.

Is	Is Not
$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

Elementary Row Operations

We defined row equivalence via the three elementary row operations. We'll use the following convenient notation:

- Swap rows i and j:
$$Ri \leftrightarrow Rj$$
- Scale row i by k:
$$kRi \to Ri$$
- Replace row j with the sum of itself and k times row i:
$$kRi + Rj \to Rj$$

We will obtain row echelon forms (ref) and reduced row echelon forms (rref) using these row operations.

Pivots

Theorem: The reduced row echelon form of a matrix is unique.

This allows the following unambiguous definition:

Definition: A **pivot position** in a matrix A is a location that corresponds to a leading 1 in the reduced echelon form of A. A **pivot column** is a column of A that contains a pivot position.

Identify the pivot position and columns given...

A — pivot positions

$$\begin{bmatrix} 0 & -3 & -6 & 4 & 9 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 1 & 4 & 5 & -9 & -7 \end{bmatrix}$$

pivot columns are 1, 2, and 4

rref of A — leading 1's

$$\begin{bmatrix} 1 & 0 & -3 & 0 & 5 \\ 0 & 1 & 2 & 0 & -3 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

pivot columns

Row Reduction Algorithm

To obtain an echelon form, we work from left to right beginning with the top row working downward.

$$\begin{bmatrix} 0 & 3 & -6 & 4 & 6 \\ 3 & -7 & 8 & 8 & -5 \\ 3 & -9 & 12 & 6 & -9 \end{bmatrix} \qquad (R_1 \leftrightarrow R_3)$$

$R_1 \leftrightarrow R_3$

$$\begin{bmatrix} 3 & -9 & 12 & 6 & -9 \\ 3 & -7 & 8 & 8 & -5 \\ 0 & 3 & -6 & 4 & 6 \end{bmatrix}$$

Step 1: The left most column is a pivot column. The top position is a pivot position. Get a nonzero entry in the top left position by row swapping if needed.

Row Reduction Algorithm

$$\begin{bmatrix} 3 & -9 & 12 & 6 & -9 \\ 3 & -7 & 8 & 8 & -5 \\ 0 & 3 & -6 & 4 & 6 \end{bmatrix}$$

$-R_1 + R_2 \to R_2$

$$\begin{bmatrix} 3 & -9 & 12 & 6 & -9 \\ 0 & 2 & -4 & 2 & 4 \\ 0 & 3 & -6 & 4 & 6 \end{bmatrix}$$

Step 2: Use row operations to get zeros in all entries below the pivot.

Row Reduction Algorithm

$$\begin{bmatrix} 3 & -9 & 12 & 6 & -9 \\ 0 & 2 & -4 & 2 & 4 \\ 0 & 3 & -6 & 4 & 6 \end{bmatrix}$$

$\frac{1}{2}R_2 \to R_2$

$$\begin{bmatrix} 3 & -9 & 12 & 6 & -9 \\ 0 & 1 & -2 & 1 & 2 \\ 0 & 3 & -6 & 4 & 6 \end{bmatrix}$$

Scratch

$$\begin{matrix} 0 & -3 & 6 & -3 & -6 \\ 0 & 3 & -6 & 4 & 6 \end{matrix}$$

$-3R_2 + R_3 \to R_3$

$$\begin{bmatrix} 3 & -9 & 12 & 6 & -9 \\ 0 & 1 & -2 & 1 & 2 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

Step 3: Ignore the row with a pivot, all rows above it, the pivot column, and all columns to its left, and repeat steps 1-2.

Row Reduction Algorithm

$$\begin{bmatrix} 3 & -9 & 12 & 6 & -9 \\ 0 & 1 & -2 & 1 & 2 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

is an echelon form.

Row Reduction Algorithm

To obtain a reduced row echelon form:

Step 4: Starting with the right most pivot and working up and to the left, use row operations to get a zero in each position above a pivot. Scale to make each pivot a 1.

$$\begin{bmatrix} 3 & -9 & 12 & 6 & -9 \\ 0 & 1 & -2 & 1 & 2 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

$\frac{1}{3} R_1 \to R_1$

$$\begin{bmatrix} 1 & -3 & 4 & 2 & -3 \\ 0 & 1 & -2 & 1 & 2 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

$-R_3 + R_2 \to R_2$

$-2R_3 + R_1 \to R_1$

Row Reduction Algorithm

$$\begin{bmatrix} 1 & -3 & 4 & 0 & -3 \\ 0 & 1 & -2 & 0 & 2 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

$3R_2 + R_1 \to R_1$

$$\begin{matrix} 1 & -3 & 4 & 0 & -3 \\ 0 & 3 & -6 & 0 & 6 \end{matrix}$$

$$\begin{bmatrix} 1 & 0 & -2 & 0 & 3 \\ 0 & 1 & -2 & 0 & 2 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

pivot columns are 1, 2 and 4

Complete Row Reduction isn't needed to find Pivots

Find the pivot positions and pivot columns of the matrix

$$\begin{bmatrix} 1 & 1 & 4 \\ -2 & 1 & -2 \\ 1 & 0 & 2 \end{bmatrix}$$

Leading entries in an ref tell us where leading 1's are in the rref.

pivot positions

pivot columns are 1 and 2.

This matrix has an ref and rref

$$\begin{bmatrix} 1 & 1 & 4 \\ 0 & 3 & 6 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}, \quad \text{respectively.}$$

Echelon Form & Solving a System

Remark: The row operations used to get an rref correspond to an **equivalent** system!

Consider the reduced echelon matrix, and describe the solution set for the associated system of equations (the one who'd have this as its augmented matrix).

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 3 \\ 0 & 0 & 1 & 0 & -2 & 4 \\ 0 & 0 & 0 & 1 & 0 & -9 \end{bmatrix}$$

$$\begin{aligned} x_1 + x_2 \qquad\qquad &= 3 \\ x_3 \quad - 2x_5 &= 4 \\ x_4 \qquad &= -9 \end{aligned}$$

This gives the solution set

$x_1 = 3 - x_2$

x_2 - free

$x_3 = 4 + 2x_5$

$x_4 = -9$

x_5 - free

Non free variables are called <u>basic</u>.

Note the basic variables are x_1, x_3, and x_4.

Consistent versus Inconsistent Systems

Consider each rref. Determine if the underlying system (the one with this as its augmented matrix) is consistent or inconsistent.

$$\begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & -3 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 2 & 3 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$x_1 + 2x_2 = 0$

$x_3 = 4$

consistent

$x_1 = 0$

$x_2 = 4$

$x_3 = -3$

consistent

$x_1 + 2x_3 = 3$

$x_2 + x_3 = 0$

$0 = 1$

always false

inconsistent.

An Existence and Uniqueness Theorem

Theorem: A linear system is consistent if and only if the right most column of the augmented matrix is **NOT** a pivot column. That is, if and only if each echelon form **DOES NOT** have a row of the form

$$[0\ 0\ \cdots\ 0\ b], \quad \text{for some nonzero } b.$$

If a linear system is consistent, then it has

(i) exactly one solution if there are no free variables, or

(ii) infinitely many solutions if there is at least one free variable.

Section 1.3: Vector Equations

Definition: A matrix that consists of one column is called a **column vector** or simply a **vector**.

e.g. $\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$

In print vectors are usually written in bold face.

In hand writing, we use an arrow over the variable, like $\vec{u}$ or $\vec{v}$

The set of vectors of the form $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ with x_1 and x_2 any real numbers is denoted by $\mathbb{R}^2$ (read "R two"). It's the set of all real ordered pairs.

Geometry

Each vector $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ corresponds to a point in the Cartesian plane. We can equate them with ordered pairs written in the traditional format $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = (x_1, x_2)$. This is **not to be confused with a row matrix.**

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \neq [x_1 \ \ x_2]$$

We can identify vectors with points or with directed line segments emanating from the origin (little arrows).

Geometry

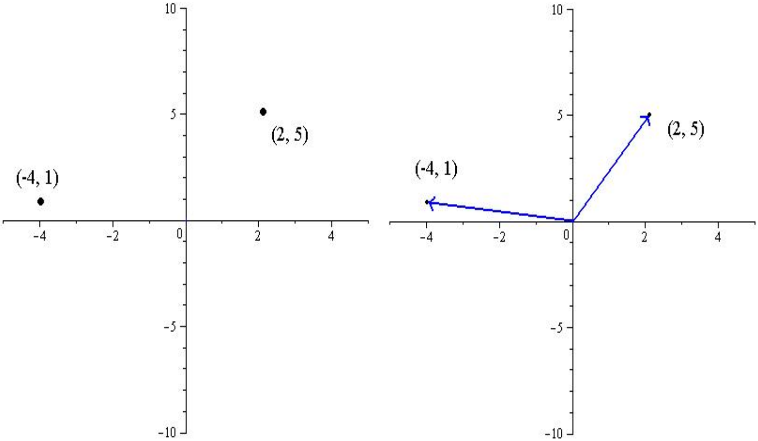

Figure: Vectors characterized as points, and vectors characterized as directed line segments.

Algebraic Operations

Let $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$, and c be a scalar[1].

Scalar Multiplication: The scalar multiple of $\mathbf{u}$

$$c\mathbf{u} = \begin{bmatrix} cu_1 \\ cu_2 \end{bmatrix}.$$

Vector Addition: The sum of vectors $\mathbf{u}$ and $\mathbf{v}$

$$\mathbf{u} + \mathbf{v} = \begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \end{bmatrix}$$

Vector Equivalence: Equality of vectors is defined by

$$\mathbf{u} = \mathbf{v} \quad \text{if and only if} \quad u_1 = v_1 \quad \text{and} \quad u_2 = v_2.$$

[1] A **scalar** is an element of the set from which u_1 and u_2 come. For our purposes, a scalar is a *real* number.

Examples

$$\mathbf{u} = \begin{bmatrix} 4 \\ -2 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} -1 \\ 7 \end{bmatrix}, \quad \text{and} \quad \mathbf{w} = \begin{bmatrix} -3 \\ \frac{3}{2} \end{bmatrix}$$

Evaluate

(a) $-2\mathbf{u}$ $= -2\begin{bmatrix} 4 \\ -2 \end{bmatrix} = \begin{bmatrix} -2\cdot 4 \\ -2\cdot(-2) \end{bmatrix} = \begin{bmatrix} -8 \\ 4 \end{bmatrix}$

(b) $-2\mathbf{u}+3\mathbf{v}$ $= -2\begin{bmatrix} 4 \\ -2 \end{bmatrix} + 3\begin{bmatrix} -1 \\ 7 \end{bmatrix} = \begin{bmatrix} -8 \\ 4 \end{bmatrix} + \begin{bmatrix} 3\cdot(-1) \\ 3\cdot 7 \end{bmatrix} = \begin{bmatrix} -8 \\ 4 \end{bmatrix} + \begin{bmatrix} -3 \\ 21 \end{bmatrix} = \begin{bmatrix} -8-3 \\ 4+21 \end{bmatrix}$

$= \begin{bmatrix} -11 \\ 25 \end{bmatrix}$

Is it true that $\mathbf{w} = -\frac{3}{4}\mathbf{u}$? $-\frac{3}{4}\vec{u} = \begin{bmatrix} -\frac{3}{4}\cdot 4 \\ -\frac{3}{4}\cdot(-2) \end{bmatrix} = \begin{bmatrix} -3 \\ \frac{3}{2} \end{bmatrix} = \vec{w}$

yes

Geometry of Algebra with Vectors

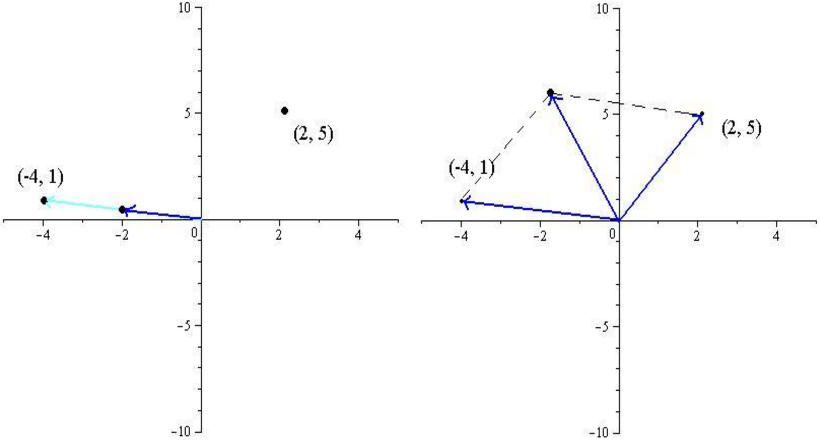

Figure: Left: $\frac{1}{2}(-4, 1) = (-2, 1/2)$. Right: $(-4, 1) + (2, 5) = (-2, 6)$

Geometry of Algebra with Vectors

Scalar Multiplication: stretches or compresses a vector but can only change direction by an angle of 0 (if $c > 0$) or π (if $c < 0$). We'll see that $0\mathbf{u} = (0, 0)$ for any vector $\mathbf{u}$.