August 19 Math 1190 sec. 52 Fall 2016

Section 1.1: Limits of Functions Using Numerical and Graphical Techniques

In *Calculus*, we consider the way in which quantities **change**. In particular, if we have a function representing some process (motion of a particle, growth of a population, spread of a disease), we can analyze it to determine the nature of how it changes. We can also use knowledge of change to reconstruct a function describing a process.

Central to analyzing change and reconstructing functions is notion of a **limit**.

The Tangent Line Problem

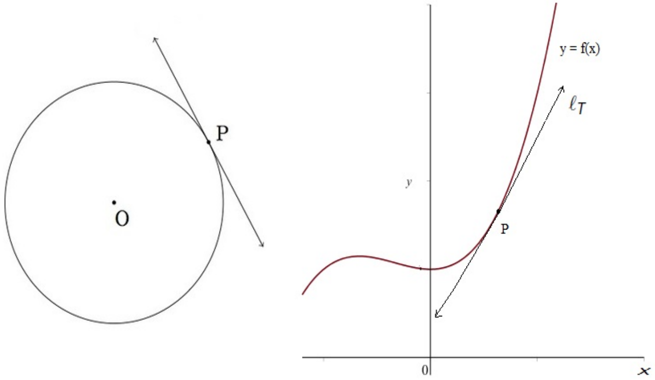


Figure: We begin by considering the tangent line problem. For a circle, a tangent line at point P is defined as the line having exactly one point in common with the circle. For the graph of a function $y = f(x)$, we define the tangent line at the point P has the line that shares the point P and has the same *slope* as the graph of f at P.

Slope of the Tangent Line

Question: What is meant by the *slope* of the function at the point P?

For now, let's assume that the graph if reasonably *nice* like the one in the figure. Let P be at $x = c$ and $y = f(c)$

$$\text{i.e.} \quad P = (c, f(c)).$$

To find a slope, we require two points. So let's take another point Q on the graph of f. In term of coordinates

$$Q = (x, f(x)).$$

The line through the two points P and Q on the graph is called a **Secant Line**. We will denote the slopes of the tangent line and the secant line as

$$m_{tan} \quad \text{and} \quad m_{sec}.$$

Slope of the Tangent Line

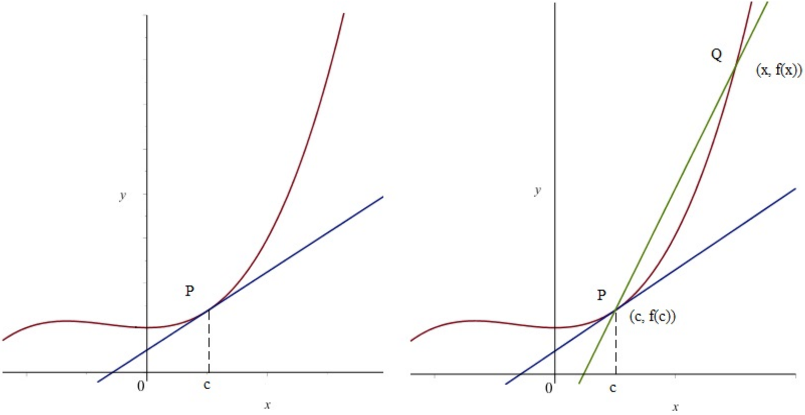
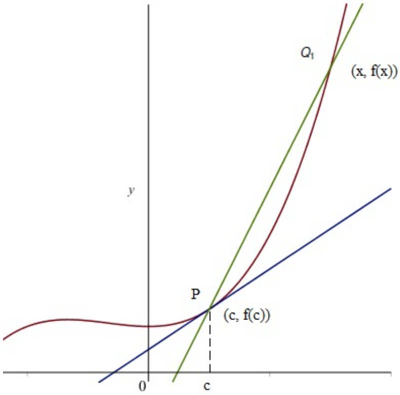


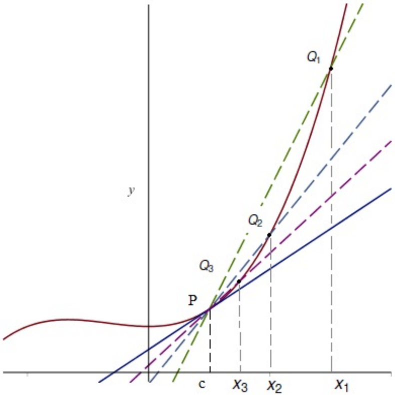
Figure: The slope of the line through P and Q (rise over run) is

$$m_{sec} = \frac{f(x) - f(c)}{x - c}$$

Slope of the Tangent Line

We consider a sequence of points $Q_1 = (x_1, f(x_1))$, $Q_2 = (x_2, f(x_2))$, and so forth in such a way that the x-values are getting closer to c. Note that the resulting secant lines tend to have slopes closer to that of the tangent line.





Slope of the Tangent Line

We call this process a *limit*. We will define the slope of the tangent line as

$$m_{tan} = \left[\text{Limit of} \quad \frac{f(x) - f(c)}{x - c} \quad \text{as } x \text{ gets closer to } c\right].$$

Our notation for this process will be

$$m_{tan} = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}.$$

The notation $\lim_{x \to c}$ reads as "the limit as x approaches c."

Notation: The notation $\lim_{x \to c}$ is always followed by an algebraic expression. It is never immediately followed by an equal sign.

A Working Definition of a Limit

Definition: Let f be defined on an open interval containing the number c except possibly at c. Then

$$\lim_{x \to c} f(x) = L$$

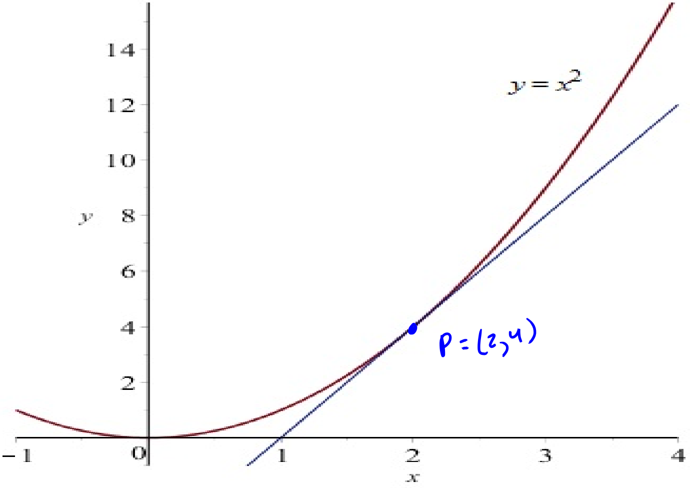
provided the value of $f(x)$ can be made arbitrarily close to the number L by taking x sufficiently close to c but not equal to c.

f(x) is a y-value, so L is associated with y-values

c is associated with x (i.e. input) values

Example

Use a calculator to determine the slope of the line tangent to the graph of $y = x^2$ at the point $(2, 4)$.



$$m_{tan} = \lim_{x \to c} \frac{f(x) - f(c)}{x - c} = \lim_{x \to 2} \frac{x^2 - 2^2}{x - 2}$$

Here

$f(x) = x^2$

and $c = 2$

$f(c) = f(2) = 2^2$

x	$\frac{f(x)-f(2)}{x-2}$
1.9	3.9
1.99	3.99
1.999	3.999
2	undefined
2.001	4.001
2.01	4.01
2.1	4.1

The table suggests that

$$m_{tan} = \lim_{x \to 2} \frac{x^2 - 2^2}{x - 2} = 4$$

Example

Use a calculator and table of values to investigate

$$\lim_{x \to 0} \frac{e^x - 1}{x}$$

$f(x) = \frac{e^x - 1}{x}$ and $c = 0$

x	$f(x) = \frac{e^x-1}{x}$
-0.1	0.9516
-0.01	0.9950
-0.001	0.9995
0	undefined
0.001	1.0005
0.01	1.0050
0.1	1.0517

The table suggests that

$$\lim_{x \to 0} \frac{e^x - 1}{x} = 1$$

Question

True or False: In order to evaluate $\lim_{x \to c} f(x)$, the value of $f(c)$ must be defined (i.e. c must be in the domain of f)?

False, by definition of the limit.

Left and Right Hand Limits

In our examples, we considered x-values to the left (less than) and to the right (greater than) c. This illustrates the notion of **one sided limits**. We have a special notation for this.

Left Hand Limit: We write

$$\lim_{x \to c^-} f(x) = L_L$$

and say *the limit as x approaches c from the left of $f(x)$ equals L_L provided we can make $f(x)$ arbitrarily close to the number L_L by taking x sufficiently close to, but less than c.*

Left and Right Hand Limits

Right Hand Limit: We write

$$\lim_{x \to c^+} f(x) = L_R$$

and say *the limit as* x *approaches* c *from the right of* $f(x)$ *equals* L_R *provided we can make* $f(x)$ *arbitrarily close to the number* L_R *by taking* x *sufficiently close to, but greater than* c.

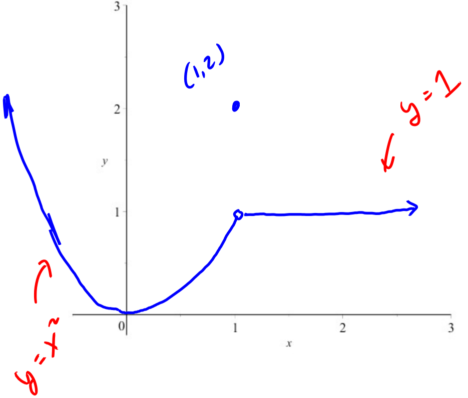
Some other common phrases:

”from the left” is the same as ”from below”

”from the right” is the same as ”from above.”

Example

Plot the function $f(x) = \begin{cases} x^2, & x < 1 \\ 2, & x = 1 \\ 1, & x > 1 \end{cases}$ Investigate $\lim_{x \to 1} f(x)$ using the graph.



x	$f(x)$
0.9	0.81
0.99	0.9801
0.999	0.9980
1	2
1.001	1
1.01	1
1.1	1

Example Continued

$$f(x) = \begin{cases} x^2, & x < 1 \\ 2, & x = 1 \\ 1, & x > 1 \end{cases}$$

$\lim_{x \to 1} f(x) =$ 1 based on the table

Based on the table

$\lim_{x \to 1^+} f(x) = 1$

$\lim_{x \to 1^-} f(x) = 1$