August 19 Math 2306 sec 51 Fall 2015

Section1.1: Definitions and Terminology

Recall:

- An *ordinary differential equation* (ODE) has one independent variable. A *partial differential equation* (PDE) has two or more.
- An **independent** variable is one derivatives are taken with respect to.
- A **dependent** variable is one derivatives are taken of.
- The **order** of an equation is the order of the highest order of derivative appearing in the equation.
- We may classify equations as being **linear** or **nonlinear**.

Classifications

Linearity: An n^{th} order differential equation is said to be **linear** if it can be written in the form

$$a_n(x)\frac{d^ny}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x).$$

Note that each of the coefficients $a_0, \ldots, a_n$ and the right hand side g may depend on the independent variable but not on the dependent variable or any of its derivatives.

Some tell-tale signs of being nonlinear are dependent variable(s) to powers other than 1, products of dependent variables, and dependent variables inside functions like trigonometric, exponential, or logs.

Examples: Classifying Equations

Identify the independent and dependent variables. Determine the order of the equation. State whether it is linear or nonlinear.

Dependent is y Independent is t

(a) $y''+2ty' = \cos t+y$

2nd order eqn.

$y''+2ty'-y = \cos t$

This is linear

(b) $$\frac{d^3y}{dx^3}+2y\frac{dy}{dx}=\frac{d^2y}{dx^2}+\tan(x)$$

Dependent is y

Independent is x

3^{rd} order

$$\frac{d^3y}{dx^3}-\frac{d^2y}{dx^2}+\underbrace{2y\frac{dy}{dx}}_{\text{nonlinear term}}=\tan x$$

coefficient of $\frac{dy}{dx}$ depends on y

The equation is nonlinear.

(c) $\ddot{\theta} + \frac{g}{\ell}\sin\theta = 0$ g and ℓ are constant

dependent is θ independent is t

recall $\ddot{\theta} = \frac{d^2\theta}{dt^2}$

nonlinear term the θ inside the sine

2^{nd} order eqn.

The eqn. is nonlinear.

Solution of $F(x, y, y', \ldots, y^{(n)}) = 0$ (*)

Definition: A function ϕ defined on an interval I[1] and possessing at least n continuous derivatives on I is a **solution** of (*) on I if upon substitution (i.e. setting $y = \phi(x)$) the equation reduces to an identity.

Definition: An **implicit solution** of (*) is a relation $G(x, y) = 0$ provided there exists at least one function $y = \phi$ that satisfies both the differential equation (*) and this relation.

[1] The interval is called the *domain of the solution* or the *interval of definition*.

Examples:

Verify that the given function is a solution of the ODE on the indicated interval.

$$\phi(t) = 3e^{2t}, \quad I = (-\infty, \infty), \quad \frac{d^2y}{dt^2} - \frac{dy}{dt} - 2y = 0$$

ϕ has derivatives of all orders on I

Set $y = \phi(t) = 3e^{2t}$, $\frac{dy}{dt} = 6e^{2t}$ and $\frac{d^2y}{dt^2} = 12e^{2t}$

$$\frac{d^2y}{dt^2} - \frac{dy}{dt} - 2y = 12e^{2t} - 6e^{2t} - 2(3e^{2t})$$

on identity $\downarrow$

$$0 = (12 - 6 - 6)e^{2t} = 0 \Rightarrow 0 = 0$$

ϕ is a solution

$$\phi(x) = 5\tan(5x), \quad I = \left(-\frac{\pi}{10}, \frac{\pi}{10}\right), \quad y' - 25 = y^2$$

Note tan(5x) is continuously differentiable

if $\quad -\frac{\pi}{2} < 5x < \frac{\pi}{2}$

i.e. $\quad -\frac{\pi}{10} < x < \frac{\pi}{10}$

so ϕ' exists and is continuous on I

Set $\quad y = \phi(x) = 5\tan(5x)$

$$y' = 5\sec^2(5x)\cdot 5 = 25\sec^2(5x)$$

$$y' - 25 = 25\sec^2(5x) - 25 = 25\left(\sec^2(5x) - 1\right)$$

$$y^2 = (5\tan(5x))^2 = 25\tan^2(5x)$$

use the ID $\sec^2\theta - 1 = \tan^2\theta$

So

$$y' - 25 = 25\left(\sec^2(5x) - 1\right) = 25\tan^2(5x) = y^2$$

we've obtained

$$25\tan^2(5x) = 25\tan^2(5x)$$

which is an identity.

ϕ is a solution on I.

$$\phi(x) = \sqrt{\ln x + 1}, \quad I = (1, \infty), \quad dx - 2xy\, dy = 0$$

We need $\ln x + 1 \geq 0 \Rightarrow \ln x \geq -1$

This holds since $\ln x > 0$ for $x > 1$.

Set $y = \phi(x) = (\ln x + 1)^{1/2}$

$$\frac{dy}{dx} = \frac{1}{2}(\ln x + 1)^{-1/2} \cdot \frac{1}{x} = \frac{1}{2\sqrt{\ln x + 1}\,(x)}$$

From $dx - 2xy\, dy = 0$ $\quad 2xy\, dy = dx$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{1}{2xy}$$ (This is the ODE in Normal form)

from above

$$\frac{dy}{dx} = \frac{1}{2x\sqrt{\ln x+1}} \quad \text{and} \quad \frac{1}{2xy} = \frac{1}{2x\sqrt{\ln x+1}}$$

So we have

$$\frac{1}{2x\sqrt{\ln x+1}} = \frac{1}{2x\sqrt{\ln x+1}}$$

an identity

So ϕ is a solution on I