August 19 Math 2306 sec 54 Fall 2015

Section1.1: Definitions and Terminology

Recall:

- An *ordinary differential equation* (ODE) has one independent variable. A *partial differential equation* (PDE) has two or more.
- An **independent** variable is one derivatives are taken with respect to.
- A **dependent** variable is one derivatives are taken of.
- The **order** of an equation is the order of the highest order of derivative appearing in the equation.

Notations and Symbols

An n^{th} order ODE, with independent variable x and dependent variable y can always be expressed as an equation

$$F(x, y, y', \ldots, y^{(n)}) = 0$$

where F is some real valued function of $n + 2$ variables.

Normal Form: If it is possible to isolate the highest derivative term, then we can write a **normal form** of the equation

$$\frac{d^n y}{dx^n} = f(x, y, y', \ldots, y^{(n-1)}).$$

Notations and Symbols

If $n = 1$ or $n = 2$, an equation in normal form would look like

$$\frac{dy}{dx} = f(x, y) \quad \text{or} \quad \frac{d^2y}{dx^2} = f(x, y, y').$$

Differential Form: A first order equation may appear in the form

$$M(x, y)\,dx + N(x, y)\,dy = 0$$

Consider y-dependent

$$N(x,y)\,dy = -M(x,y)\,dx$$

$$\frac{dy}{dx} = \frac{-M(x,y)}{N(x,y)}$$

for $N(x,y) \neq 0$

Consider x dependent

$$M(x,y)\,dx = -N(x,y)\,dy$$

$$\frac{dx}{dy} = \frac{-N(x,y)}{M(x,y)}$$ for $M(x,y) \neq 0$

Classifications

Linearity: An n^{th} order differential equation is said to be **linear** if it can be written in the form

$$a_n(x)\frac{d^ny}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x).$$

Note that each of the coefficients $a_0, \ldots, a_n$ and the right hand side g may depend on the independent variable but not on the dependent variable or any of its derivatives.

Some tell-tale signs of being nonlinear are dependent variable(s) to powers other than 1, products of dependent variables, and dependent variables inside functions like trigonometric, exponential, or logs.

Examples (Linear -vs- Nonlinear)

Both linear

$$y'' + 4y = 0$$

$$t^2 \frac{d^2x}{dt^2} + 2t\frac{dx}{dt} - x = e^t$$

looks like

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = g(x)$$

where

$a_2(x) = 1$

$a_1(x) = 0$

$a_0(x) = 4$

$g(x) = 0$

Looks like

$$a_2(t)x'' + a_1(t)x' + a_0(t)x = g(t)$$

where

$a_2(t) = t^2$

$a_1(t) = 2t$

$a_0(t) = -1$

$g(t) = e^t$

Examples (Linear -vs- Nonlinear)

Both non linear

$$\frac{d^3y}{dx^3} + \underbrace{\left(\frac{dy}{dx}\right)^4}_{= \left(\frac{dy}{dx}\right)^3 \frac{dy}{dx}} = x^3$$

power other than 1 on $\frac{dy}{dx}$

non linear term

the eqn is non linear

$$u'' + u' = \cos u$$

non linear term

dependent variable inside the cosine

The equation is nonlinear

Identify the independent and dependent variables. Determine the order of the equation. State whether it is linear or nonlinear.

(a) $y'' + 2ty' = \cos t + y$

Dependent is y

Independent is t

2^{nd} order equation

$y'' + 2ty' - y = \cos t$

This is linear.

(b) $\dfrac{d^3y}{dx^3}+2y\dfrac{dy}{dx}=\dfrac{d^2y}{dx^2}+\tan(x)$

Dependent is y

Independent is x

3^{rd} order

$$\frac{d^3y}{dx^3}-\frac{d^2y}{dx^2}+\underbrace{2y\frac{dy}{dx}}_{\text{nonlinear term}}=\tan x$$

The equation is nonlinear.

(c) $\ddot{\theta} + \frac{g}{\ell} \sin\theta = 0$ g and ℓ are constant

Dependent is θ

Independent is t

recall $\ddot{\theta} = \frac{d^2\theta}{dt^2}$

2^{nd} order

$\sin\theta$ is a nonlinear term since θ is dependent

The eqn. is nonlinear.

Solution of $F(x, y, y', \dots, y^{(n)}) = 0$ (*)

Definition: A function ϕ defined on an interval I[1] and possessing at least n continuous derivatives on I is a **solution** of (*) on I if upon substitution (i.e. setting $y = \phi(x)$) the equation reduces to an identity.

Definition: An **implicit solution** of (*) is a relation $G(x, y) = 0$ provided there exists at least one function $y = \phi$ that satisfies both the differential equation (*) and this relation.

[1] The interval is called the *domain of the solution* or the *interval of definition*.

Examples:

Verify that the given function is a solution of the ODE on the indicated interval.

$$\phi(t) = 3e^{2t}, \quad I = (-\infty, \infty)\,, \quad \frac{d^2y}{dt^2} - \frac{dy}{dt} - 2y = 0$$

Note $\phi(t)$ has continuous derivatives of all orders on I.

Set $y = \phi(t) = 3e^{2t}$

$$\frac{dy}{dt} = 6e^{2t} \quad \text{and} \quad \frac{d^2y}{dt^2} = 12e^{2t}$$

$$\frac{d^2y}{dt^2} - \frac{dy}{dt} - 2y = 12e^{2t} - 6e^{2t} - 2\left(3e^{2t}\right)$$

$$0 = e^{2t}(12-6-6) = 0$$

$0 = 0$ is an identity.

ϕ is a solution on I.

$$\phi(x) = 5\tan(5x), \quad I = \left(-\frac{\pi}{10}, \frac{\pi}{10}\right), \quad y' - 25 = y^2$$

Recall $\tan\theta$ is continuous and differentiable

on $\left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$.

If $\frac{-\pi}{2} < 5x < \frac{\pi}{2}$ then $\frac{-\pi}{10} < x < \frac{\pi}{10}$

so ϕ is continuously differentiable on I.

Set $y = \phi(x) = 5\tan(5x)$

$y' = 5\sec^2(5x) \cdot 5 = 25\sec^2(5x)$

$$y' - 25 = y^2$$

$$y' - 25 = 25\sec^2(5x) - 25 = 25(\sec^2(5x) - 1)$$

$$y^2 = (5\tan(5x))^2 = 25\tan^2(5x)$$

Recall $\sec^2\theta - 1 = \tan^2\theta$

Hence $y' - 25 = 25(\underbrace{\sec^2(5x) - 1}_{\text{Identity from trigonometry}}) = 25\tan^2(5x) = y^2$

ϕ is a solution on I.