August 20 Math 2306 sec. 53 Fall 2018

Section 2: Initial Value Problems

An initial value problem consists of an ODE with a certain type of additional conditions.

Solve the equation [1]

$$\frac{d^n y}{dx^n} = f(x, y, y', \ldots, y^{(n-1)}) \tag{1}$$

subject to the *initial conditions*

$$y(x_0) = y_0, \quad y'(x_0) = y_1, \quad \ldots, y^{(n-1)}(x_0) = y_{n-1}. \tag{2}$$

The problem (1)–(2) is called an *initial value problem* (IVP).

[1] on some interval I containing x_0.

Example

Part 1

Show that for any constant c the relation $x^2 + y^2 = c$ is an implicit solution of the ODE

$$\frac{dy}{dx} = -\frac{x}{y}$$

Using implicit diff. assume $x^2+y^2=C$ for some C.

$2x + 2y\frac{dy}{dx} = 0$ solve for $\frac{dy}{dx}$

$2y\frac{dy}{dx} = -2x \Rightarrow$ for $y \neq 0$ $\frac{dy}{dx} = \frac{-2x}{2y} = \frac{-x}{y}$

This is the correct ODE. So $x^2+y^2=C$ defines solutions to $\frac{dy}{dx} = \frac{-x}{y}$.

Example

Part 2

Use the preceding results to find an explicit solution of the IVP

$$\frac{dy}{dx} = -\frac{x}{y}, \quad y(0) = -2$$

From befor, solutions are given by

$x^2 + y^2 = C$ for constant C.

If we impose $y(0) = -2$, we get

$(0)^2 + (-2)^2 = C \Rightarrow C = 4$

So to satisfy the IVP, we require

$$x^2 + y^2 = 4.$$

To get an explicit solution, solve for y:

$$y^2 = 4 - x^2 \implies y = \sqrt{4-x^2} \quad \text{or}$$
$$y = -\sqrt{4-x^2}.$$

Recall that $y(0) = -2$. Only the 2nd option satisfies the sign requirement.

$$\boxed{\text{The solution to the IVP is } \quad y = -\sqrt{4-x^2}.}$$

Graphical Interpretation

$y = y_0 e^{-x^2}$

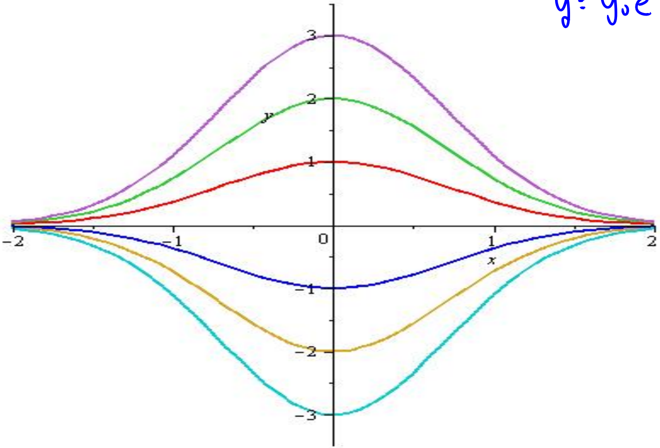

Figure: Each curve solves $y' + 2xy = 0$, $y(0) = y_0$. Each colored curve corresponds to a different value of y_0

Existence and Uniqueness

Two important questions we can always pose (and sometimes answer) are

(1) Does an IVP have a solution? (existence) and

(2) If it does, is there just one? (uniqueness)

Hopefully it's obvious that we can't solve $\underbrace{\left(\frac{dy}{dx}\right)^2 + 1}_{\geq 1} = \underbrace{-y^2}_{\leq 0}.$

Uniqueness

Consider the IVP

$$\frac{dy}{dx} = x\sqrt{y} \quad y(0) = 0$$

Verify that $y = \frac{x^4}{16}$ is a solution of the IVP.

Solving an IVP means satisfying a differential equation AND satisfying an initial condition.

Show that $y = \frac{x^4}{16}$ satisfies the initial condition:

we need to show $y=0$ when $x=0$.

$y(0) = \frac{0^4}{16} = \frac{0}{16} = 0$ as required

$$\frac{dy}{dx} = x\sqrt{y} \quad y(0) = 0$$

Show that $y = \frac{x^4}{16}$ solves the differential equation:

Note $y' = \frac{4x^3}{16} = \frac{x^3}{4}$ also

$$x\sqrt{y} = x\sqrt{\frac{x^4}{16}} = x\,\frac{\sqrt{x^4}}{\sqrt{16}} = x\,\frac{|x^2|}{4} = x\,\frac{x^2}{4} = \frac{x^3}{4}$$

$$\frac{dy}{dx} = \frac{x^3}{4} = \frac{x^3}{4} = x\sqrt{y}$$

the function satisfies the ODE

$$\frac{dy}{dx} = x\sqrt{y} \quad y(0) = 0$$

this IVP has at least 2 solns.

$y = \frac{x^4}{16}$ and $y = 0$

Verify that the function $y = 0$ is a solution to the IVP.

Letting $y=0$, note $y(0)=0$ it solves the I.C.

Also, if $y(x)=0$ then

$\frac{dy}{dx} = 0$ and $x\sqrt{y} = x\sqrt{0} = 0$

$\frac{dy}{dx} = 0 = x\sqrt{y}$ it solves the ODE too!

Section 3: Separation of Variables

The simplest type of equation we could encounter would be of the form

$$\frac{dy}{dx} = g(x).$$

For example, solve the ODE

$$\frac{dy}{dx} = 4e^{2x} + 1.$$

Integrate

$$\int \frac{dy}{dx}\,dx = \int (4e^{2x} + 1)\,dx$$

$$y = 2e^{2x} + x + C$$

Separable Equations

Definition: The first order equation $y' = f(x, y)$ is said to be **separable** if the right side has the form

$$f(x, y) = g(x)h(y).$$

That is, a separable equation is one that has the form

$$\frac{dy}{dx} = g(x)h(y).$$

Determine which (if any) of the following are separable.

(a) $\frac{dy}{dx} = x^3 y$

yes in fact $g(x) = x^3$ and $h(y) = y$

(b) $\frac{dy}{dx} = 2x + y$

Not separable

(c) $\dfrac{dy}{dx} = \sin(xy^2)$ Not separable

(d) $\dfrac{dy}{dt} - te^{t-y} = 0$ $\Rightarrow$ $\dfrac{dy}{dt} = te^{t-y} = te^{t} \cdot e^{-y}$

Is separable w/

$g(t) = te^{t}$ and $h(y) = e^{-y}$