August 21 Math 2306 sec 51 Fall 2015

Section1.1: Definitions and Terminology

Solution of $F(x, y, y', \ldots, y^{(n)}) = 0$ (*)

Definition: A function ϕ defined on an interval I[1] and possessing at least *n* continuous derivatives on *I* is a **solution** of (*) on *I* if upon substitution (i.e. setting $y = \phi(x)$) the equation reduces to an identity.

Definition: An **implicit solution** of (*) is a relation $G(x, y) = 0$ provided there exists at least one function $y = \phi$ that satisfies both the differential equation (*) and this relation.

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[1]The interval is called the *domain of the solution* or the *interval of definition*.

Example

Verify that the relation defines and implicit solution of the differential equation.

$$y^2 - 2x^2y = 1, \quad (y - x^2)\frac{dy}{dx} = 2xy$$

Find $\frac{dy}{dx}$ using implicit differentiation.

$$2y\frac{dy}{dx} - 4xy - 2x^2\frac{dy}{dx} = 0$$

$$y\frac{dy}{dx} - 2xy - x^2\frac{dy}{dx} = 0$$

$$(y - x^2)\frac{dy}{dx} - 2xy = 0$$

$$(y - x^2) \frac{dy}{dx} = 2xy$$

which is the diff. eqn given.

Function vs Solution

The interval of defintion has to be an interval.

Consider $y' = -y^2$. Clearly $y = \frac{1}{x}$ solves the DE. The interval of defintion can be $(-\infty, 0)$, or $(0, \infty)$—or any interval that doesn't contain the origin. But it can't be $(-\infty, 0) \cup (0, \infty)$ because this isn't an interval!

Often, we'll take *I* to be the largest, or one of the largest, possible interval. It may depend on other information.
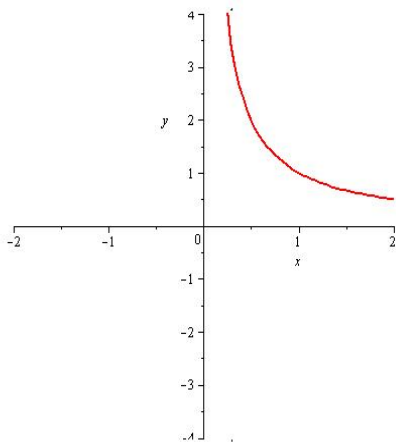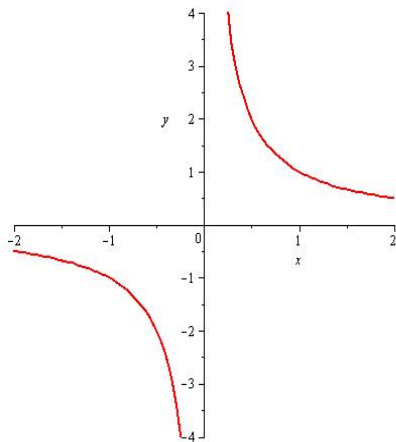
Figure: Left: Plot of $f(x) = \frac{1}{x}$ as a **function**. Right: Plot of $f(x) = \frac{1}{x}$ as a possible **solution** of an ODE.

Show that for any choice of constants c_1 and c_2, $y = c_1 x + \frac{c_2}{x}$ is a solution of the differential equation on $I = (0, 1)$.

$$x^2 y'' + xy' - y = 0$$

$y = c_1 x + \dfrac{c_2}{x}$ has   2 continuous derivatives on $(0,1)$.

$y' = c_1 - \dfrac{c_2}{x^2}$, $y'' = \dfrac{2c_2}{x^3}$

$$x^2 \left(\frac{2c_2}{x^3} \right) + x \left(c_1 - \frac{c_2}{x^2} \right) - \left(c_1 x + \frac{c_2}{x} \right) \stackrel{?}{=} 0$$

$$\frac{2c_2}{x} + c_1 x - \frac{c_2}{x} - c_1 x - \frac{c_2}{x} \overset{?}{=} 0$$

$$\frac{1}{x}\left(2c_2 - c_2 - c_2\right) + x\left(c_1 - c_1\right) \overset{?}{=} 0$$

$$0 \quad + \quad 0 \quad = 0$$

Some Terms

- A **parameter** is an unspecified constant such as c_1 and c_2 in the last example.

- A **family of solutions** is a collection of solution functions that only differ by a parameter.

- An *n***-parameter family of solutions** is one containing *n* parameters (e.g. $c_1 x + \frac{c_2}{x}$ is a 2 parameter family).

- A **particular solution** is one with no arbitrary constants in it.

- The **trivial solution** is the simple constant function $y = 0$.

- An **integral curve** is the graph of one solution (perhaps from a family).

Section 1.2: Initial Value Problems

An initial value problem consists of an ODE with additional conditions.

Solve the equation [2]

$$\frac{d^n y}{dx^n} = f(x, y, y', \ldots, y^{(n-1)}) \tag{1}$$

subject to the *initial conditions*

$$y(x_0) = y_0, \quad y'(x_0) = y_1, \quad \ldots, y^{(n-1)}(x_0) = y_{n-1}. \tag{2}$$

The problem (1)–(2) is called an *initial value problem* (IVP).

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[2] on some interval *I* containing x_0.

First order case:

$$\frac{dy}{dx} = f(x, y), \quad y(x_0) = y_0$$

rate of change
of y is $f(x,y)$

$y = y_0$ and when $x = x_0$

Second order case:

$$\frac{d^2y}{dx^2} = f(x, y, y'), \quad y(x_0) = y_0, \quad y'(x_0) = y_1$$

graph
passes through
(x_0, y_0)

graph has
slope y_1
at (x_0, y_0)

Example

Given that $y = c_1 x + \frac{c_2}{x}$ is a 2-parameter family of solutions of $x^2 y'' + xy' - y = 0$, solve the IVP

$$x^2 y'' + xy' - y = 0, \quad y(1) = 1, \quad y'(1) = 3$$

we know that the soln. to the ODE looks

like $\quad y = c_1 x + \frac{c_2}{x}$. we need to satisfy

$y(1) = 1 \quad$ and $\quad y'(1) = 3$.

$$y' = c_1 - \frac{c_2}{x^2}$$

$$y(1) = C_1 \cdot 1 + \frac{C_2}{1} = 1 \implies C_1 + C_2 = 1$$

$$y'(1) = C_1 - \frac{C_2}{1^2} = 3 \implies C_1 - C_2 = 3$$

add eqns

$$2C_1 = 4 \implies C_1 = 2$$

subtract eqns

$$2C_2 = -2 \implies C_2 = -1$$

so the member of the family that solves the initial conditions has $C_1 = 2$, $C_2 = -1$

The solution to the IVP is

$$y = 2x - \frac{1}{x} .$$