August 21 Math 2306 sec 54 Fall 2015

Section1.1: Definitions and Terminology

Solution of $F(x, y, y', \ldots, y^{(n)}) = 0$ (*)

Definition: A function ϕ defined on an interval I[1] and possessing at least n continuous derivatives on I is a **solution** of (*) on I if upon substitution (i.e. setting $y = \phi(x)$) the equation reduces to an identity.

Definition: An **implicit solution** of (*) is a relation $G(x, y) = 0$ provided there exists at least one function $y = \phi$ that satisfies both the differential equation (*) and this relation.

[1] The interval is called the *domain of the solution* or the *interval of definition*.

Examples:

Verify that the relation defines and implicit solution of the differential equation.

$$y^2 - 2x^2y = 1, \quad (y - x^2)\frac{dy}{dx} = 2xy$$

$y^2 - 2x^2y - 1 = 0$

We'll use implicit differentiation to show that if y satisfies the relation, then $\frac{dy}{dx}$ satisfies the diff. equation.

$y^2 - 2x^2y = 1$ take der. w/ respect to X

$$2y\frac{dy}{dx} - 2\left(2xy + x^2\frac{dy}{dx}\right) = 0$$

$$y\frac{dy}{dx} - \left(2xy + x^2\frac{dy}{dx}\right) = 0$$

$$y\frac{dy}{dx} - 2xy - x^2\frac{dy}{dx} = 0$$

$$(y-x^2)\frac{dy}{dx} - 2xy = 0 \implies \underbrace{(y-x^2)\frac{dy}{dx} = 2xy}_{\text{The given ODE}}$$

Hence the relation defines a solution of the ODE.

Function vs Solution

The interval of defintion has to be an interval.

Consider $y' = -y^2$. Clearly $y = \frac{1}{x}$ solves the DE. The interval of defintion can be $(-\infty, 0)$, or $(0, \infty)$—or any interval that doesn't contain the origin. But it can't be $(-\infty, 0) \cup (0, \infty)$ because this isn't an interval!

Often, we'll take I to be the largest, or one of the largest, possible interval. It may depend on other information.

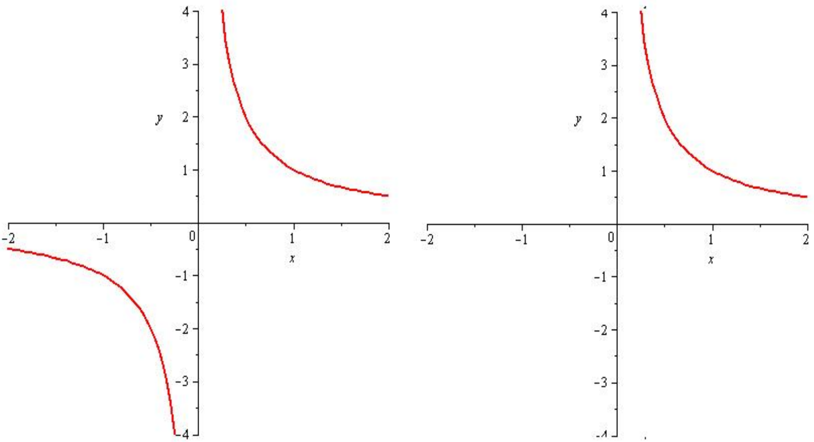

Figure: Left: Plot of $f(x) = \frac{1}{x}$ as a **function**. Right: Plot of $f(x) = \frac{1}{x}$ as a possible **solution** of an ODE.

Show that for any choice of constants c_1 and c_2, $y = c_1 x + \frac{c_2}{x}$ is a solution of the differential equation on $I = (0, 1)$.

$$x^2 y'' + xy' - y = 0$$

y is continuous and differentiable on I with derivatives of all orders.

$$y = c_1 x + \frac{c_2}{x}, \quad y' = c_1 - \frac{c_2}{x^2}, \quad y'' = \frac{2c_2}{x^3}$$

$$x^2\left(\frac{2c_2}{x^3}\right) + x\left(c_1 - \frac{c_2}{x^2}\right) - \left(c_1 x + \frac{c_2}{x}\right) \stackrel{?}{=} 0$$

$$\frac{2c_2}{x} + c_1 x - \frac{c_2}{x} - c_1 x - \frac{c_2}{x} \stackrel{?}{=} 0$$

$$\frac{1}{x}\left(2c_2 - c_2 - c_2\right) + x\left(c_1 - c_1\right) = 0$$

$$0 + 0 = 0$$

Some Terms

- A **parameter** is an unspecified constant such as c_1 and c_2 in the last example.
- A **family of solutions** is a collection of solution functions that only differ by a parameter.
- An ***n*-parameter family of solutions** is one containing n parameters (e.g. $c_1 x + \frac{c_2}{x}$ is a 2 parameter family).
- A **particular solution** is one with no arbitrary constants in it.
- The **trivial solution** is the simple constant function $y = 0$.
- An **integral curve** is the graph of one solution (perhaps from a family).

Section 1.2: Initial Value Problems

An initial value problem consists of an ODE with additional conditions.

Solve the equation [2]

$$\frac{d^n y}{dx^n} = f(x, y, y', \ldots, y^{(n-1)}) \tag{1}$$

subject to the *initial conditions*

$$y(x_0) = y_0, \quad y'(x_0) = y_1, \quad \ldots, y^{(n-1)}(x_0) = y_{n-1}. \tag{2}$$

The problem (1)–(2) is called an *initial value problem* (IVP).

[2] on some interval I containing x_0.

First order case:

$$\frac{dy}{dx} = f(x, y), \quad y(x_0) = y_0$$

$y = y_0$ when $x = x_0$

i.e., the solution passes through the point (x_0, y_0)

Second order case:

$$\frac{d^2y}{dx^2} = f(x, y, y'), \quad y(x_0) = y_0, \quad y'(x_0) = y_1$$

curve passes through (x_0, y_0)

curve has slope y_1 at (x_0, y_0)