August 22 MATH 1113 sec. 51 Fall 2018

Section 2.2: The Algebra of Functions

Let f and g be functions, and suppose that x is in the domain of each. Then define $f+g$, $f-g$, fg and f/g, and use the following notation

- $(f+g)(x) = f(x) + g(x)$
- $(f-g)(x) = f(x) - g(x)$
- $(fg)(x) = f(x)g(x)$
- $\left(\dfrac{f}{g}\right)(x) = \dfrac{f(x)}{g(x)}$ provided $g(x) \neq 0$

Domain

For functions f and g, the domain of $f+g$, $f-g$, and fg is the set of all x such that x is in the domain of f and x is in the domain of g.

the intersection of the domains of f and g

The domain of f/g is the set of all all x such that x is in the domain of f, x is in the domain of g, and $g(x) \neq 0$.

the domain of fg less any x such that $g(x) = 0$

Example

Let $f(x) = \sqrt{x+1}$ and $g(x) = 3x^2$. Evaluate

(a) $(f+g)(3)$ $= f(3) + g(3) = \sqrt{3+1} + 3(3^2) = \sqrt{4} + 3(9) = 2 + 27$
$= 29$

(b) $(f-g)\left(-\frac{3}{4}\right)$

$= f\left(\frac{-3}{4}\right) - g\left(\frac{-3}{4}\right)$

$= \sqrt{\frac{-3}{4}+1} - 3\left(\frac{-3}{4}\right)^2 = \sqrt{\frac{1}{4}} - 3\left(\frac{9}{16}\right)$

$= \frac{1}{2} - \frac{27}{16} = \frac{8-27}{16} = \frac{-19}{16}$

Example

Let $f(x) = \sqrt{x+1}$ and $g(x) = 3x^2$. Identify the domains of $f+g$ and f/g.

For x in the domain of f, we require

$x+1 \geq 0 \Rightarrow x \geq -1$. In interval notation

the domain of f is $[-1, \infty)$. The domain

of g is all reals, i.e. $(-\infty, \infty)$. The

intersection is $[-1, \infty)$. The

domain of $f+g$ is $[-1, \infty)$.

$f(x) = \sqrt{x+1}$, $g(x) = 3x^2$

The domain of $f+g$ is $[-1, \infty)$.

Note $g(x) = 0 \Rightarrow 3x^2 = 0 \Rightarrow x = 0$.

The domain of $\dfrac{f}{g}$ is

$[-1, 0) \cup (0, \infty)$.

Question

Let $f(x) = 2x^2$ and $g(x) = \dfrac{2}{x-5}$. Evaluate $(fg)(2)$ and $\left(\dfrac{f}{g}\right)(1)$.

(a) $(fg)(2) = -3$ and $\left(\dfrac{f}{g}\right)(1) = -4$

(b) $(fg)(2) = -\frac{16}{3}$ and $\left(\dfrac{f}{g}\right)(1) = -1$

(c) $(fg)(2) = -\frac{16}{3}$ and $\left(\dfrac{f}{g}\right)(1) = -4$

(d) $(fg)(2) = \frac{16}{3}$ and $\left(\dfrac{f}{g}\right)(1) = -1$

$$(fg)(2) = f(2)g(2)$$
$$= 2(2^2)\left(\frac{2}{2-5}\right)$$
$$= 2\cdot 4\left(\frac{2}{-3}\right) = \frac{-16}{3}$$

$$\left(\frac{f}{g}\right)(1) = \frac{f(1)}{g(1)} = \frac{2(1^2)}{\frac{2}{1-5}}$$
$$= \frac{2}{\frac{-2}{4}} = 2(-2) = -4$$

Section 2.3: Compositions

Suppose a spherical balloon is inflated so that the radius after time t seconds is given by the function $r(t) = 2t$ cm. The volume of a sphere of radius r is known to be $V(r) = \frac{4}{3}\pi r^3$. Note that

- r is a function of t, and
- V is a function of r, making
- V a function of t (through its dependence on r). In fact,

$$V(t) = V(r(t)) = \frac{4}{3}\pi(2t)^3 = \frac{32}{3}\pi t^3.$$

This is an example of a **composition** of functions.

Composition: Definition and Notation

Let f and g be functions. Then the **composite** function denoted

$$f \circ g,$$

also called the **composition** of f and g, is defined by

$$(f \circ g)(x) = f(g(x)).$$

The domain of $f \circ g$ is the set of all x in the domain of g such that $g(x)$ is in the domain of f.

The expression $f \circ g$ is read "f composed with g", and $(f \circ g)(x)$ is read "f of g of x".

Example

Let $f(x) = \sqrt{x-1}$ and $g(x) = \frac{2}{x+1}$. Evaluate each expression if possible.

(a) $(f \circ g)(1) = f(g(1)) = f\left(\frac{2}{1+1}\right) = f\left(\frac{2}{2}\right) = f(1) = \sqrt{1-1} = 0$

(b) $(f \circ g)(0) = f(g(0)) = f\left(\frac{2}{0+1}\right) = f(2) = \sqrt{2-1} = \sqrt{1} = 1$

(c) $(g \circ f)(0) = g(f(0)) = g(\sqrt{0-1}) = g(\sqrt{-1})$ undefined

0 is not in the domain of $g \circ f$

it's not in the domain of f either.

Question

Let $f(x) = \sqrt{x-1}$ and $g(x) = \frac{2}{x+1}$. Evaluate $(g \circ f)(1)$ if possible.

(a) $(g \circ f)(1) = 0$

(b) $(g \circ f)(1) = 1$

(c) $(g \circ f)(1) = 2$

(d) $(g \circ f)(1)$ is undefined

$= g(f(1))$

$= g(\sqrt{1-1})$

$= g(0) = \frac{2}{0+1} = 2$