August 22 Math 1190 sec. 52 Fall 2016

Section 1.1: Limits of Functions Using Numerical and Graphical Techniques

Definition: Let f be defined on an open interval containing the number c except possibly at c. Then

$$\lim_{x\to c} f(x) = L$$

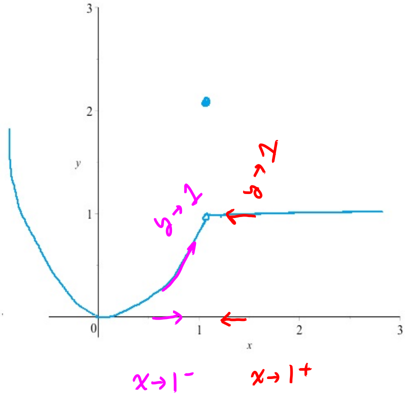
provided the value of $f(x)$ can be made arbitrarily close to the number L by taking x sufficiently close to c but not equal to c.

We similarly defined left and right hand limits using the conventional notation.

$$\text{left} \quad \lim_{x\to c^-} f(x) = L_L, \quad \text{right} \quad \lim_{x\to c^+} f(x) = L_R$$

Example (from Friday)

Plot the function $f(x) = \begin{cases} x^2, & x < 1 \\ 2, & x = 1 \\ 1, & x > 1 \end{cases}$ Investigate $\lim_{x \to 1} f(x)$ using the graph.



x	$f(x)$
0.9	0.81
0.99	0.9801
0.999	0.9981
1	2
1.001	1
1.01	1
1.1	1

Example Continued

$$f(x) = \begin{cases} x^2, & x < 1 \\ 2, & x = 1 \\ 1, & x > 1 \end{cases}$$

Based on the table, we concluded that

$$\lim_{x \to 1^+} f(x) = 1 \quad \text{and} \quad \lim_{x \to 1^-} f(x) = 1 \quad \text{and} \quad \lim_{x \to 1} f(x) = 1$$

Both 1-sided limits are 1. All y-values tend toward the value $y=1$ as $x \to 1$.

Note $f(1)=2$, but this doesn't affect the limit.

Observations

Observation 1: The limit L of a function $f(x)$ as x approaches c does not depend on whether $f(c)$ exists or what it's value may be.

Observation 2: If $\lim_{x \to c} f(x) = L$, then the number L is unique. That is, a function can not have two different limits as x approaches a single number c.

Observation 3: A function need not have a limit as x approaches c. If $f(x)$ can not be made arbitrarily close to any one number L as x approaches c, then we say that $\lim_{x \to c} f(x)$ **does not exist** (shorthand **DNE**).

Questions

(1) **True or False** It is possible that both $\lim_{x \to 3} f(x) = 5$ AND $f(3) = 7$.

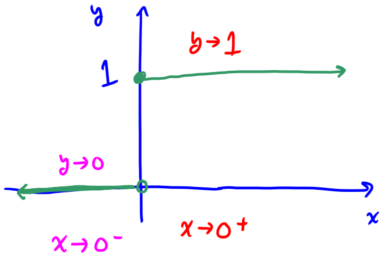
(2) **True or False** It is possible that both $\lim_{x \to 3} f(x) = 5$ AND $\lim_{x \to 3} f(x) = 7$.

A Limit Failing to Exist

Consider $H(x) = \begin{cases} 0, & x < 0 \\ 1, & x \geq 0 \end{cases}$. Evaluate if possible

$$\lim_{x\to 0^-} H(x), \quad \lim_{x\to 0^+} H(x), \quad \text{and} \quad \lim_{x\to 0} H(x)$$

H is called the Heaviside step function.



$$H(x) = \begin{cases} 0, & x < 0 \\ 1, & x \geq 0 \end{cases}$$

$\lim_{x \to 0^-} H(x) = 0$

$\lim_{x \to 0^+} H(x) = 1$

$\lim_{x \to 0} H(x)$ DNE

The function has a jump at 0.
The left and right limits don't agree.

Weakness of Technology

Suppose we wish to investigate

$$\lim_{x \to 0} \sin\left(\frac{\pi}{x^2}\right).$$

We consider values of x closer to zero, and plug them into a calculator. Let's look at two attempts.

x	$\sin\left(\frac{\pi}{x^2}\right)$
-0.1	0
-0.01	0
-0.001	0
0	undefined
0.001	0
0.01	0
0.1	0

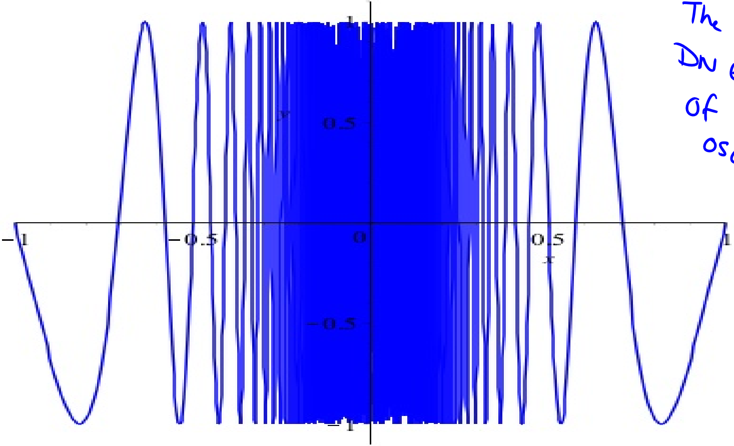
Suggests the limit is zero.

x	$\sin\left(\frac{\pi}{x^2}\right)$
$-\frac{2}{3}$	0.707
$-\frac{2}{13}$	0.707
$-\frac{2}{23}$	0.707
0	undefined
$\frac{2}{23}$	0.707
$\frac{2}{13}$	0.707
$\frac{2}{3}$	0.707

Suggests the limit is 0.707...

Weakness of Technology

In every interval containing zero, the graph of $\sin(\pi/x^2)$ passes through every y-value between -1 and 1 infinitely many times.



The limit DNE because of oscillations

Figure: $y = \sin\left(\frac{\pi}{x^2}\right)$

Evaluating Limits

As this example illustrates, we would like to avoid too much reliance on technology for evaluating limits. The next section will be devoted to techniques for doing this for reasonably well behaved functions. We close with one theorem.

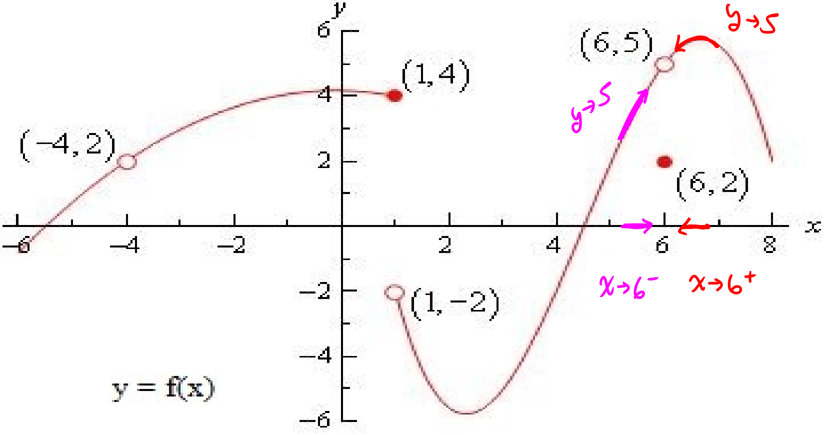
Theorem: Let f be defined on an open interval containing c except possible at c. Then

$$\lim_{x \to c} f(x) = L$$

if and only if

$$\lim_{x \to c^-} f(x) = L \quad \text{and} \quad \lim_{x \to c^+} f(x) = L.$$

Example



Use the previous figure to evaluate if possible

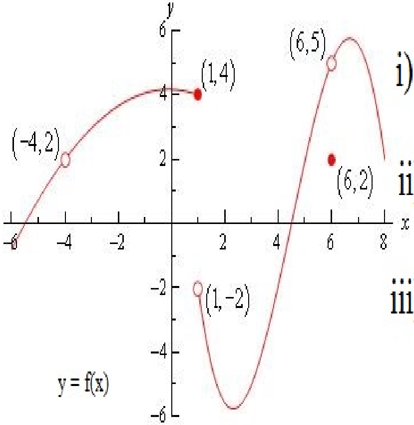
(a) $\lim_{x \to 6^-} f(x) = 5$

(b) $\lim_{x \to 6^+} f(x) = 5$

(c) $f(6) = 2$, the point $(6,2)$ is on the graph.

(d) $\lim_{x \to 6} f(x) = 5$

Question

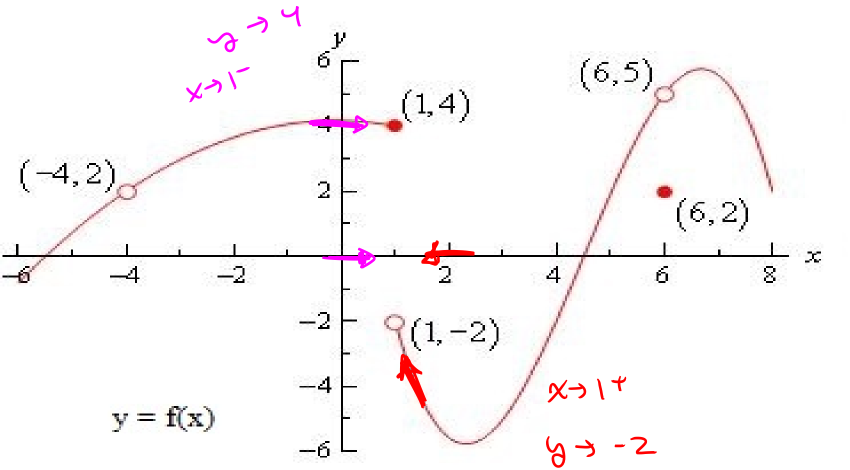


i) $\lim_{x \to 1^-} f(x) =$ (a) 1 (b) 4 (c) -2 (d) DNE

ii) $\lim_{x \to 1^+} f(x) =$ (a) 1 (b) 4 (c) -2 (d) DNE

iii) $\lim_{x \to 1} f(x) =$ (a) 1 (b) 4 (c) -2 (d) DNE

Limits and Jumps



Section 1.2: Limits of Functions Using Properties of Limits

We begin with two of the simplest limits we may encounter.

Theorem: If $f(x) = A$ where A is a constant, then for any real number c

$$\lim_{x \to c} f(x) = \lim_{x \to c} A = A$$

The limit of a constant is that constant.

Theorem: If $f(x) = x$, then for any real number c

$$\lim_{x \to c} f(x) = \lim_{x \to c} x = c$$

Examples

(a) $\lim_{x\to 0} 7 = 7$

1st theorem with $A=7$ and $c=0$

(b) $\lim_{x\to \pi} 3\pi = 3\pi$

1st thm with $A=3\pi$, $c=\pi$

(c) $\lim_{x\to -\sqrt{5}} x = -\sqrt{5}$

2nd thm. with $c=-\sqrt{5}$

By 2nd thm $\lim_{x\to 4} x = 4$ so

(d) $\lim_{x\to 4^-} x = 4$

$\lim_{x\to 4^-} x = 4$ and $\lim_{x\to 4^+} x = 4$.

Additional Limit Law Theorems

Suppose

$$\lim_{x \to c} f(x) = L, \quad \lim_{x \to c} g(x) = M, \quad \text{and } k \text{ is constant.}$$

Theorem: (Sums) $\lim_{x \to c}(f(x)+g(x)) = L+M$

Theorem: (Differences) $\lim_{x \to c}(f(x)-g(x)) = L-M$

Theorem: (Constant Multiples) $\lim_{x \to c} kf(x) = kL$

Theorem: (Products) $\lim_{x \to c} f(x)g(x) = LM$

Examples

Use the limit law theorems to evaluate if possible

(a) $\lim_{x \to 2} (x^2 + 3x)$

Note $\lim_{x \to 2} x = 2$ and $\lim_{x \to 2} 3 = 3$

$$\lim_{x \to 2} (x^2 + 3x) = \lim_{x \to 2} x^2 + \lim_{x \to 2} 3x \quad \text{sum}$$

$$= \left(\lim_{x \to 2} x\right) \cdot \left(\lim_{x \to 2} x\right) + \left(\lim_{x \to 2} 3\right) \cdot \left(\lim_{x \to 2} x\right) \quad \text{product}$$

$$= 2 \cdot 2 + 3 \cdot 2 = 4 + 6 = 10$$