August 22 Math 2306 sec. 53 Fall 2018

Section 3: Separation of Variables

In this section, we will restrict attention to **first order, separable** differential equations. This is the first section in which we will start with an ODE and use an approach to find a solution.

Definition: A **separable** differential equation is a first order equation that can be written in the form

$$\frac{dy}{dx} = g(x)h(y).$$

The defining characteristic is that the right hand side can be expressed as a product of a function of x alone and a function of y alone.

Solving Separable Equations

Recall that from $\frac{dy}{dx} = g(x)$, we can integrate both sides

$$\int \frac{dy}{dx}\,dx = \int g(x)\,dx.$$

$$\int dy = \int g(x)\,dx$$

$$y = G(x) + C$$

where $G(x)$ is any antiderivative of $g(x)$

We'll use this observation!

Solving Separable Equations

Let's assume that it's safe to divide by $h(y)$ and let's set $p(y) = 1/h(y)$. We solve (usually find an implicit solution) by **separating the variables.**

Let's set y's on one side, x's on the other.

$$\frac{dy}{dx} = g(x)h(y)$$

$$\frac{1}{h(y)}\frac{dy}{dx} = g(x)$$

multiply by dx

$$* \quad \frac{dy}{dx}dx = dy$$

$$\frac{1}{h(y)}\frac{dy}{dx}dx = g(x)dx$$

$$p(y)\,dy = g(x)dx$$

$$\int p(y)dy = \int g(x)dx$$

$$P(y) = G(x) + C$$

where P, G are antider. of p and g.

Solve the ODE

It is separable.

$$\frac{dy}{dx} = -\frac{x}{y} = -x\left(\frac{1}{y}\right)$$

$$\frac{1}{1/y}\frac{dy}{dx} = -x$$

$$y\frac{dy}{dx} = -x \implies \int y\frac{dy}{dx}\,dx = \int -x\,dx$$

$$\int y\,dy = -\int x\,dx \implies \frac{1}{2}y^2 = -\frac{1}{2}x^2 + k$$

Cleaning it up $\quad y^2 = -x^2 + 2k \quad$ let $C = 2k$

$$y^2 = -x^2 + C \implies x^2 + y^2 = C$$

$$\boxed{\text{The solutions are defined implicitly by the relation } x^2+y^2 = C.}$$

An Initial Value Problem: Solve the IVP

$$te^{t-y}\,dt - dy = 0, \qquad y(0) = 1$$

The ODE is separable.

$$dy = te^{t-y}dt = te^{t}e^{-y}dt \Rightarrow \frac{1}{e^{-y}}dy = te^{t}dt$$

$$\int e^{y}dy = \int te^{t}dt$$

using Int by parts on the right

$u = t \qquad du = dt$

$v = e^{t} \qquad dv = e^{t}dt$

$$e^{y} = te^{t} - \int e^{t}dt$$

$$= te^{t} - e^{t} + C$$

Solutions to the ODE are given by

$$e^y = te^t - e^t + C$$

Now we impose the condition $y(0) = 1$.

$y=1$ when $t=0$

$$e^1 = 0e^0 - e^0 + C \implies e = 0 - 1 + C \implies C = e+1$$

The solution to the IVP is given implicitly by

$$\boxed{e^y = te^t - e^t + e + 1}.$$

The explicit solution is $y = \ln\left(te^t - e^t + e + 1\right)$

Caveat regarding division by $h(y)$.

Recall that the IVP $\quad \dfrac{dy}{dx} = x\sqrt{y}, \quad y(0) = 0$

has two solutions

$$y(x) = \frac{x^4}{16} \quad \text{and} \quad y(x) = 0.$$

If we separate the variables

$$\frac{1}{\sqrt{y}}\,dy = x\,dx$$

we lose the second solution.

Why? Dividing out $\sqrt{y}$ assumes $y \neq 0$!

Solutions Defined by Integrals

Recall (Fundamental Theorem of Calculus) for g continuous on an interval containing x_0 and x

$$\frac{d}{dx}\int_{x_0}^{x} g(t)\,dt = g(x) \quad \text{and} \quad \int_{x_0}^{x} \frac{dy}{dt}\,dt = y(x) - y(x_0).$$

Assuming g is continuous at x_0, use this to solve

$$\frac{dy}{dx} = g(x), \quad y(x_0) = y_0.$$

Mult. by dt and integrate from x_0 to x

(using the dummy variable t)

$$\int_{x_0}^{x} \frac{dy}{dt}\,dt = \int_{x_0}^{x} g(t)\,dt$$

$$y(x) - y(x_0) = \int_{x_0}^{x} g(t)\,dt \qquad y(x_0) = y_0$$

$$\boxed{y(x) = y_0 + \int_{x_0}^{x} g(t)\,dt}$$

exercise: verify that this solves the IVP

Example

Express the solution of the IVP in terms of an integral.

$$\frac{dy}{dx} = \sin(x^2), \quad y(\sqrt{\pi}) = 1$$

here $g(x) = \sin(x^2)$ and

$x_0 = \sqrt{\pi}$ and $y_0 = 1$

The form was

$$y(x) = y_0 + \int_{x_0}^{x} g(t)\,dt$$

So

$$\boxed{y(x) = 1 + \int_{\sqrt{\pi}}^{x} \sin(t^2)\,dt}$$