August 22 Math 2306 sec. 56 Fall 2017

Section 3: Separation of Variables

Definition: The first order equation $y' = f(x,y)$ is said to be **separable** if the right side has the form

$$f(x,y) = g(x)h(y).$$

That is, a separable equation is one that has the form

$$\frac{dy}{dx} = g(x)h(y).$$

Determine which (if any) of the following are separable.

(c) $\dfrac{dy}{dx} = \sin(xy^2)$

Not separable

xy^2 would be, but $\sin(xy^2)$ is not

(d) $\dfrac{dy}{dt} - te^{t-y} = 0$ $\Rightarrow$ $\dfrac{dy}{dt} = te^{t-y} = \underbrace{te^{t}}_{g(t)} \cdot \underbrace{e^{-y}}_{h(y)}$

It is separable.

Solving Separable Equations

Recall that from $\frac{dy}{dx} = g(x)$, we can integrate both sides

$$\int \frac{dy}{dx}\, dx = \int g(x)\, dx.$$

$$\Rightarrow \int dy = \int g(x)\, dx$$

$$y = G(x) + C$$

for some antiderivative G of g

We'll use this observation!

Solving Separable Equations

Let's assume that it's safe to divide by $h(y)$ and let's set $p(y) = 1/h(y)$. We solve (usually find an implicit solution) by **separating the variables.**

$$\frac{dy}{dx} = g(x)h(y)$$

Divide by $h(y)$

$$\frac{1}{h(y)}\frac{dy}{dx} = g(x)$$

multiply by dx and Integrate

$$\int \frac{1}{h(y)}\frac{dy}{dx}\,dx = \int g(x)\,dx$$

$$\frac{dy}{dx}dx = dy$$

$$\int p(y)\,dy = \int g(x)\,dx$$

$$P(y) = G(x) + C$$

where P and G are antiderivatives of p and g, respectively.

$P(y) = G(x) + C$ defines a one-parameter family of solutions implicitly.

Solve the ODE

$$\frac{dy}{dx} = -\frac{x}{y} = -x \cdot \frac{1}{y} \Rightarrow \frac{1}{1/y}\frac{dy}{dx} = -x$$

$$y\frac{dy}{dx} = -x$$

$$\int y \frac{dy}{dx}\, dx = \int -x\, dx$$

$$\int y\, dy = -\int x\, dx \Rightarrow \frac{y^2}{2} = \frac{-x^2}{2} + C$$

mult by 2, let $2C = k$

$$y^2 = -x^2 + k \Longrightarrow x^2 + y^2 = k$$

An IVP[1]

$$\frac{dy}{dt} - te^{t-y} = 0, \qquad y(0) = 1$$

$$\frac{dy}{dt} = te^t \cdot e^{-y}$$

$$\frac{1}{e^{-y}} \frac{dy}{dt} = te^t \quad \Rightarrow \quad e^y \frac{dy}{dt} = te^t$$

$$\int e^y \frac{dy}{dt}\, dt = \int te^t\, dt$$

$$\int e^y\, dy = \int te^t\, dt$$

$$e^y = te^t - \int e^t\, dt$$

Int by parts

$u = t \quad du = dt$

$v = e^t \quad dv = e^t dt$

[1] Recall IVP stands for *initial value problem*.

$$e^y = te^t - e^t + C$$ This family solves the ODE

Impose $y(0)=1$ $\Rightarrow$ when $t=0$, $y=1$

$$e^1 = 0e^0 - e^0 + C \Rightarrow e = -1 + C \Rightarrow C = e+1$$

The solution to the IVP is given by

$$\boxed{e^y = te^t - e^t + e + 1.}$$

Caveat regarding division by $h(y)$.

Recalll that the IVP $\quad \dfrac{dy}{dx} = x\sqrt{y}, \quad y(0) = 0$

has two solutions

$$y(x) = \frac{x^4}{16} \quad \text{and} \quad y(x) = 0.$$

If we separate the variables

$$\frac{1}{\sqrt{y}}\,dy = x\,dx$$

we lose the second solution.

Why? When $\sqrt{y}$ was divided, it was tacitly assumed that it was nonzero.

Solutions Defined by Integrals

Recall (Fundamental Theorem of Calculus)

$F(b) - F(a)$

$$\frac{d}{dx}\int_{x_0}^{x} g(t)\,dt = g(x) \quad \text{and} \quad \int_{x_0}^{x} \frac{dy}{dt}\,dt = y(x) - y(x_0).$$

Use this to solve

$$\frac{dy}{dx} = g(x), \quad y(x_0) = y_0$$

Start with $\quad \frac{dy}{dt} = g(t)$

Integrate from x_0 to x $\quad \int_{x_0}^{x} \frac{dy}{dt}\,dt = \int_{x_0}^{x} g(t)\,dt$

$$y(x) - y(x_0) = \int_{x_0}^{x} g(t)\,dt$$

$$y(x) = y(x_0) + \int_{x_0}^{x} g(t)\,dt$$

$$y(x) = y_0 + \int_{x_0}^{x} g(t)\,dt$$ ← solution to the IVP

Let's verify:

Initial condition $\quad y(x_0) = y_0 + \int_{x_0}^{x_0} g(t)\,dt$

$y(x_0) = y_0$ $\quad$ Satisfies the IC

ODE: $\quad \frac{d}{dx}y(x) = \frac{d}{dx}\left(y_0 + \int_{x_0}^{x} g(t)\,dt\right)$

$$\frac{dy}{dx} = \frac{d}{dx}y_0 + \frac{d}{dx}\int_{x_0}^{x} g(t)\,dt$$

$$\frac{dy}{dx} = g(x)$$ solves the ODE too!

Example: Express the solution of the IVP in terms of an integral.

$$\frac{dy}{dx} = \sin(x^2), \quad y(\sqrt{\pi}) = 1$$

here $g(t) = \sin(t^2)$, $x_0 = \sqrt{\pi}$ and $y_0 = 1$

$$y(x) = y_0 + \int_{x_0}^{x} g(t)\,dt$$

So $$\boxed{y(x) = 1 + \int_{\sqrt{\pi}}^{x} \sin(t^2)\,dt}$$

Section 4: First Order Equations: Linear

A first order linear equation has the form

$$a_1(x)\frac{dy}{dx} + a_0(x)y = g(x).$$

If $g(x) = 0$ the equation is called **homogeneous.** Otherwise it is called **nonhomogeneous.**

Provided $a_1(x) \neq 0$ on the interval I of definition of a solution, we can write the **standard form** of the equation

$$P(x) = \frac{a_0(x)}{a_1(x)}$$

$$\frac{dy}{dx} + P(x)y = f(x). \qquad f(x) = \frac{g(x)}{a_1(x)}$$

We'll be interested in equations (and intervals I) for which P and f are continuous on I.

Solutions (the General Solution)

$$\frac{dy}{dx} + P(x)y = f(x).$$

It turns out the solution will always have a basic form of $y = y_c + y_p$ where

- y_c is called the **complementary** solution and would solve the problem
$$\frac{dy}{dx} + P(x)y = 0$$
(called the associated homogeneous equation), and
- y_p is called the **particular** solution, and is heavily influenced by the function $f(x)$.

The cool thing is that our solution method will get both parts in one process—we won't get this benefit with higher order equations!

Motivating Example

$$x^2\frac{dy}{dx}+2xy = e^x$$

This is not in standard form, but that's ok for now.

Note $\frac{d}{dx}\left[x^2y\right] = \underbrace{x^2\frac{dy}{dx} + 2xy}_{\text{our left side!}}$

Our eqn is $\frac{d}{dx}\left[x^2y\right] = e^x$

Integrate! $\int \frac{d}{dx}\left[x^2y\right]dx = \int e^x\,dx$

$$x^2 y = e^x + C$$

Divide out x^2 to get y!

$$y = \frac{e^x + C}{x^2}$$

We will springboard a technique based on this example.