Section 3: Separation of Variables

The simplest type of equation we could encounter would be of the form

$$\frac{dy}{dx} = g(x).$$

For example, solve the ODE

$$\frac{dy}{dx} = 4e^{2x} + 1.$$ 

$$y = \int (4e^{2x} + 1) \, dx$$

$$= 2e^{2x} + x + C$$

$$\star \int e^{ax} \, dx = \frac{1}{a}e^{ax} + C$$ 
for $a \neq 0$
Definition: The first order equation \( y' = f(x, y) \) is said to be **separable** if the right side has the form

\[
f(x, y) = g(x)h(y).
\]

That is, a separable equation is one that has the form

\[
\frac{dy}{dx} = g(x)h(y).
\]
Determine which (if any) of the following are separable.

(a) \( \frac{dy}{dx} = x^3 y \)

is separable with
\[ g(x) = x^3 \quad \text{and} \quad h(y) = y \]

(b) \( \frac{dy}{dx} = 2x + y \)

this is not separable.
(c) \[ \frac{dy}{dx} = \sin(xy^2) \] not separable, \( xy^2 \) would be but \( \sin(xy^2) \) is not

(d) \[ \frac{dy}{dt} - te^{t-y} = 0 \] \[ \implies \frac{dy}{dt} = te^{t-y} = te \cdot e^{t-y} \] \[ \Rightarrow g(t) \cdot h(y) \] This is separable.
Solving Separable Equations

Recall that from $\frac{dy}{dx} = g(x)$, we can integrate both sides

$$\int \frac{dy}{dx} \, dx = \int g(x) \, dx.$$ 

We’ll use this observation!
Solving Separable Equations

Let’s assume that it’s safe to divide by \( h(y) \) and let’s set \( p(y) = 1/h(y) \). We solve (usually find an implicit solution) by \textbf{separating the variables}.

\[
\frac{dy}{dx} = g(x)h(y)
\]

\[
\frac{1}{h(y)} \frac{dy}{dx} = g(x)
\]

\[
\int p(y) \frac{dy}{dx} \, dx = \int g(x) \, dx
\]

\[
\int p(y) \, dy = \int g(x) \, dx
\]
\[ P(y) = G(x) + C \]

where \( P \) and \( G \) are anti-derivatives of \( p \) and \( g \), respectively.

\[ P(y) = G(x) + C \] defines a 1-parameter family of solutions implicitly.
Solve the ODE

\[
\frac{dy}{dx} = -\frac{x}{y} \quad \Rightarrow \quad \frac{1}{\sqrt{y}} \frac{dy}{dx} = -x
\]

\[
y \frac{dy}{dx} = -x
\]

\[
\Rightarrow \quad \int y \frac{dy}{dx} \, dx = \int -x \, dx \quad \Rightarrow \quad \int y \, dy = -\int x \, dx
\]

\[
\Rightarrow \quad \frac{y^2}{2} = -\frac{x^2}{2} + C
\]

\[
\text{Multi by 2 and let } 2C = k
\]

\[
y^2 = -x^2 + k \quad \Rightarrow \quad x^2 + y^2 = k
\]
An IVP$^1$

\[ \frac{dy}{dt} - te^{t-y} = 0, \quad y(0) = 1 \]

\[ \frac{dy}{dt} = te^t \cdot e^y \quad \Rightarrow \quad e^{-y} \frac{dy}{dt} = te^t \]

\[ e^y \frac{dy}{dt} = te^t \quad \Rightarrow \quad \int e^y \, dy = \int te^t \, dt \]

\[ e^y = te^t - \int e^t \, dt \]

\[ \text{Int by parts} \]
\[ u = t, \quad \text{du} = dt \]
\[ v = e^t, \quad \text{dv} = e^t \, dt \]

$^1$Recall IVP stands for initial value problem.
\[ e^y = te^t - e^t + C \] 

1 parameter family of solutions to the ODE

Impose \( y(0) = 1 \) \( \Rightarrow \) when \( t = 0 \), \( y = 1 \)

\[ C^1 = 0e^0 - e^0 + C \Rightarrow e = -1 + C \Rightarrow C = e + 1 \]

The solution to the IVP is given by

\[ e^y = te^t - e^t + e + 1. \]
Caveat regarding division by $h(y)$.

Recall that the IVP \( \frac{dy}{dx} = x\sqrt{y}, \quad y(0) = 0 \)
has two solutions
\[
y(x) = \frac{x^4}{16} \quad \text{and} \quad y(x) = 0.
\]
If we separate the variables
\[
\frac{1}{\sqrt{y}} \, dy = x \, dx
\]
we lose the second solution.

Why? when dividing by \( \sqrt{y} \), we tacitly assume it's nonzero.
Solutions Defined by Integrals

Recall (Fundamental Theorem of Calculus)

\[
\frac{d}{dx} \int_{x_0}^{x} g(t) \, dt = g(x) \quad \text{and} \quad \int_{x_0}^{x} \frac{dy}{dt} \, dt = y(x) - y(x_0).
\]

Use this to solve

\[
\frac{dy}{dx} = g(x), \quad y(x_0) = y_0
\]

\[
\frac{dy}{dt} = g(t) \Rightarrow \int_{x_0}^{x} \frac{dy}{dt} \, dt = \int_{x_0}^{x} g(t) \, dt
\]

\[
\Rightarrow y(x) - y(x_0) = \int_{x_0}^{x} g(t) \, dt
\]

\[
y(x) - y_0 = \int_{x_0}^{x} g(t) \, dt
\]
\[ y(x) = y_0 + \int_{x_0}^{x} g(t) \, dt \] is a soln to the IVP.

Let's verify: Initial condition

\[ y(x_0) = y_0 + \int_{x_0}^{x_0} g(t) \, dt = y_0 + 0 = y_0 \]

\( y \) satisfies the I.C.

ODE:

\[
\frac{d}{dx} y(x) = \frac{d}{dx} \left( y_0 + \int_{x_0}^{x} g(t) \, dt \right)
\]

\[
\frac{dy}{dx} = \frac{dy}{dx} y_0 + \frac{1}{dx} \int_{x_0}^{x} g(t) \, dt
\]

\[
= 0 + \frac{1}{dx} \int_{x_0}^{x} g(t) \, dt = 0 + g(x)
\]

\[
\frac{dy}{dx} = g(x) \quad y \text{ solves the ODE too!}
\]
Example: Express the solution of the IVP in terms of an integral.

\[ \frac{dy}{dx} = \sin(x^2), \quad y(\sqrt{\pi}) = 1 \]

here \( g(t) = \sin(t^2) \)

\[ x_0 = \sqrt{\pi} \text{ and } y_0 = 1 \]

Our solution to the IVP is

\[ y(x) = y_0 + \int_{x_0}^{x} g(t) \, dt \]

\[ y(x) = 1 + \int_{\sqrt{\pi}}^{x} \sin(t^2) \, dt \]
Section 4: First Order Equations: Linear

A first order linear equation has the form

\[ a_1(x) \frac{dy}{dx} + a_0(x)y = g(x). \]

If \( g(x) = 0 \) the equation is called \textbf{homogeneous}. Otherwise it is called \textbf{nonhomogeneous}.

Provided \( a_1(x) \neq 0 \) on the interval \( I \) of definition of a solution, we can write the \textbf{standard form} of the equation

\[ \frac{dy}{dx} + P(x)y = f(x). \]

\[ P(x) = \frac{a_0(x)}{a_1(x)} \]

\[ f(x) = \frac{g(x)}{a_1(x)} \]

We’ll be interested in equations (and intervals \( I \)) for which \( P \) and \( f \) are continuous on \( I \).
Solutions (the General Solution)

\[
\frac{dy}{dx} + P(x)y = f(x).
\]

It turns out the solution will always have a basic form of \( y = y_c + y_p \) where

- \( y_c \) is called the **complementary** solution and would solve the problem
  \[
  \frac{dy}{dx} + P(x)y = 0
  \]
  (called the associated homogeneous equation), and

- \( y_p \) is called the **particular** solution, and is heavily influenced by the function \( f(x) \).

The cool thing is that our solution method will get both parts in one process—we won’t get this benefit with higher order equations!