August 22 Math 2306 sec. 57 Fall 2017

Section 3: Separation of Variables

The simplest type of equation we could encounter would be of the form

$$\frac{dy}{dx} = g(x).$$

For example, solve the ODE

$$\frac{dy}{dx} = 4e^{2x} + 1.$$

$$y = \int (4e^{2x} + 1)\, dx$$

$$= 4 \cdot \frac{1}{2} e^{2x} + x + C$$

$$\boxed{y = 2e^{2x} + x + C}$$

$$* \int e^{ax}\, dx = \frac{1}{a} e^{ax} + C$$

for $a \neq 0$

Separable Equations

Definition: The first order equation $y' = f(x, y)$ is said to be **separable** if the right side has the form

$$f(x, y) = g(x)h(y).$$

That is, a separable equation is one that has the form

$$\frac{dy}{dx} = g(x)h(y).$$

Determine which (if any) of the following are separable.

(a) $\dfrac{dy}{dx} = x^3 y$

yes it's separable with

$g(x) = x^3$ and $h(y) = y$

(b) $\dfrac{dy}{dx} = 2x + y$

this is not separable.

(c) $\frac{dy}{dx} = \sin(xy^2)$

Not separable, xy^2 would be but $\sin(xy^2)$ is not

(d) $\frac{dy}{dt} - te^{t-y} = 0$

$$\Rightarrow \quad \frac{dy}{dt} = te^{t-y} = \underbrace{te^{t}}_{g(t)} \cdot \underbrace{e^{-y}}_{h(y)}$$

This is separable.

Solving Separable Equations

Recall that from $\frac{dy}{dx} = g(x)$, we can integrate both sides

$$\int \frac{dy}{dx}\,dx = \int g(x)\,dx.$$

$$\frac{dy}{dx}\,dx = dy$$

$$\int dy = \int g(x)\,dx$$

$$y = G(x) + C$$

where G is some antiderivative of g

We'll use this observation!

Solving Separable Equations

Let's assume that it's safe to divide by $h(y)$ and let's set $p(y) = 1/h(y)$. We solve (usually find an implicit solution) by **separating the variables.**

$$\frac{dy}{dx} = g(x)h(y)$$

Divide by h(y)

$$\frac{1}{h(y)}\frac{dy}{dx} = g(x)$$

multiply by dx and integrate

$$\int p(y)\frac{dy}{dx}\,dx = \int g(x)\,dx$$

$$\int p(y)\,dy = \int g(x)\,dx$$

Integrate

$$P(y) = G(x) + C$$

where P and G are antiderivatives of p and g, respectively.

$P(y) = G(x) + C$ defines a 1-parameter family of solutions implicitly.

Solve the ODE

$$\frac{dy}{dx} = -\frac{x}{y} = -x \cdot \frac{1}{y} \Rightarrow \frac{1}{1/y}\frac{dy}{dx} = -x$$

$$y\frac{dy}{dx} = -x$$

$$\Rightarrow \int y \frac{dy}{dx}\,dx = \int -x\,dx \Rightarrow \int y\,dy = -\int x\,dx$$

$$\Rightarrow \frac{y^2}{2} = \frac{-x^2}{2} + C$$

Mult by 2 and let $2C = k$

$$y^2 = -x^2 + k \Rightarrow x^2 + y^2 = k$$

An IVP[1]

$$\frac{dy}{dt} - te^{t-y} = 0, \qquad y(0) = 1$$

$$\frac{dy}{dt} = te^t \cdot e^{-y} \Rightarrow \frac{1}{e^{-y}}\frac{dy}{dt} = te^t$$

$$e^y \frac{dy}{dt} = te^t \Rightarrow \int e^y \frac{dy}{dt}\,dt = \int te^t\,dt$$

$$\int e^y\,dy = \int te^t\,dt$$

Int by parts

$u = t \qquad du = dt$

$v = e^t \qquad dv = e^t\,dt$

$$e^y = te^t - \int e^t\,dt$$

[1]Recall IVP stands for *initial value problem*.

$e^y = te^t - e^t + C$ 1 parameter family of solutions to the ODE

Impose $y(0)=1$ $\Rightarrow$ when $t=0$, $y=1$

$e^1 = 0e^0 - e^0 + C \Rightarrow e = -1 + C \Rightarrow C = e+1$

$$\boxed{\text{The solution to the IVP is given by } \quad e^y = te^t - e^t + e + 1.}$$

Caveat regarding division by $h(y)$.

Recalll that the IVP $\quad \dfrac{dy}{dx} = x\sqrt{y}, \quad y(0) = 0$

has two solutions

$$y(x) = \frac{x^4}{16} \quad \text{and} \quad y(x) = 0.$$

If we separate the variables

$$\frac{1}{\sqrt{y}}\, dy = x\, dx$$

we lose the second solution.

Why? when dividing by $\sqrt{y}$, we tacitly assume it's nonzero.

Solutions Defined by Integrals

Recall (Fundamental Theorem of Calculus) for g cont.

$$\frac{d}{dx}\int_{x_0}^{x} g(t)\,dt = g(x) \quad \text{and} \quad \int_{x_0}^{x} \frac{dy}{dt}\,dt = y(x) - y(x_0).$$

$F(b) - F(a)$

Use this to solve

$$\frac{dy}{dx} = g(x), \quad y(x_0) = y_0$$

$$\frac{dy}{dt} = g(t) \Rightarrow \int_{x_0}^{x} \frac{dy}{dt}\,dt = \int_{x_0}^{x} g(t)\,dt$$

$$\Rightarrow \quad y(x) - y(x_0) = \int_{x_0}^{x} g(t)\,dt$$

$$y(x) - y_0 = \int_{x_0}^{x} g(t)\,dt$$

$$y(x) = y_0 + \int_{x_0}^{x} g(t)\,dt$$ is a soln to the IVP.

Let's verify: Initial condition

$$y(x_0) = y_0 + \int_{x_0}^{x_0} g(t)\,dt = y_0 + 0 = y_0$$

0

y satisfies the I.C.

ODE:

$$\frac{d}{dx} y(x) = \frac{d}{dx}\left(y_0 + \int_{x_0}^{x} g(t)\,dt \right)$$

0

$$\frac{dy}{dx} = \frac{d}{dx} y_0 + \frac{d}{dx} \int_{x_0}^{x} g(t)\,dt$$

$$= 0 + \frac{d}{dx} \int_{x_0}^{x} g(t)\,dt = 0 + g(x)$$

$$\frac{dy}{dx} = g(x)$$

y solves the ODE too!

Example: Express the solution of the IVP in terms of an integral.

$$\frac{dy}{dx} = \sin(x^2), \quad y(\sqrt{\pi}) = 1$$

here $g(t) = \sin(t^2)$

$x_0 = \sqrt{\pi}$ and $y_0 = 1$

$$y(x) = y_0 + \int_{x_0}^{x} g(t)\,dt$$

$\boxed{\text{Our solution to the IVP is} \quad y(x) = 1 + \int_{\sqrt{\pi}}^{x} \sin(t^2)\,dt}$

Section 4: First Order Equations: Linear

A first order linear equation has the form

$$a_1(x)\frac{dy}{dx} + a_0(x)y = g(x).$$

If $g(x) = 0$ the equation is called **homogeneous.** Otherwise it is called **nonhomogeneous.**

Provided $a_1(x) \neq 0$ on the interval I of definition of a solution, we can write the **standard form** of the equation

$$P(x) = \frac{a_0(x)}{a_1(x)}$$

$$\frac{dy}{dx} + P(x)y = f(x). \qquad f(x) = \frac{g(x)}{a_1(x)}$$

We'll be interested in equations (and intervals I) for which P and f are continuous on I.

Solutions (the General Solution)

$$\frac{dy}{dx} + P(x)y = f(x).$$

It turns out the solution will always have a basic form of $y = y_c + y_p$ where

- y_c is called the **complementary** solution and would solve the problem
$$\frac{dy}{dx} + P(x)y = 0$$
(called the associated homogeneous equation), and
- y_p is called the **particular** solution, and is heavily influenced by the function $f(x)$.

The cool thing is that our solution method will get both parts in one process—we won't get this benefit with higher order equations!