August 22 Math 3260 sec. 58 Fall 2017

Section 1.3: Vector Equations

Definition: A matrix that consists of one column is called a **column vector** or simply a **vector**.

In print, these are usually written in bold, e.g. **u**, to distinguish variables denoting vectors from those denoting scalars. Written by hand, we will use an over arrow, $\vec{u}$, to denote a vector.

Algebraic Operations

Let $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$, and c be a scalar.

Scalar Multiplication: The scalar multiple of $\mathbf{u}$

$$c\mathbf{u} = \begin{bmatrix} cu_1 \\ cu_2 \end{bmatrix}.$$

Vector Addition: The sum of vectors $\mathbf{u}$ and $\mathbf{v}$

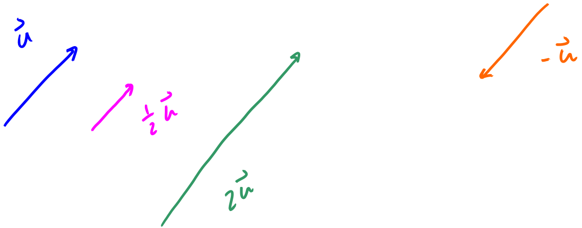
$$\mathbf{u} + \mathbf{v} = \begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \end{bmatrix}$$

Vector Equivalence: Equality of vectors is defined by

$$\mathbf{u} = \mathbf{v} \quad \text{if and only if} \quad u_1 = v_1 \quad \text{and} \quad u_2 = v_2.$$

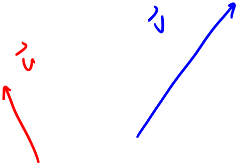
Geometry of Algebra with Vectors

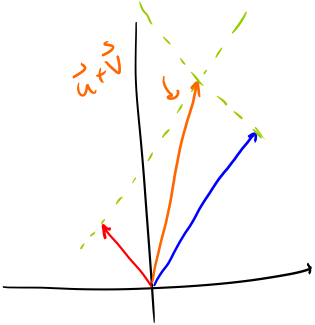
Scalar Multiplication: stretches or compresses a vector but can only change direction by an angle of 0 (if $c > 0$) or π (if $c < 0$). We'll see that $0\mathbf{u} = (0, 0)$ for any vector $\mathbf{u}$.



Geometry of Algebra with Vectors

Vector Addition: The sum $\mathbf{u} + \mathbf{v}$ of two vectors (each different from $(0,0)$) is the the fourth vertex of a parallelogram whose other three vertices are (u_1, u_2), (v_1, v_2), and $(0,0)$.





Vectors in $\mathbb{R}^n$

A vector in $\mathbb{R}^3$ is a 3×1 column matrix. These are ordered triples. For example

$$\mathbf{a} = \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix}, \quad \text{or} \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}.$$

A vector in $\mathbb{R}^n$ for $n \geq 2$ is a $n \times 1$ column matrix. These are ordered n-tuples. For example

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}.$$

The Zero Vector: is the vector whose entries are all zeros. It will be denoted by $\mathbf{0}$ or $\vec{0}$ and is not to be confused with the scalar 0.

Algebraic Properties on $\mathbb{R}^n$

For every $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ in $\mathbb{R}^n$ and scalars c and d[1]

(i) $\mathbf{u}+\mathbf{v}=\mathbf{v}+\mathbf{u}$

(ii) $(\mathbf{u}+\mathbf{v})+\mathbf{w}=\mathbf{u}+(\mathbf{v}+\mathbf{w})$

(iii) $\mathbf{u}+\mathbf{0}=\mathbf{0}+\mathbf{u}=\mathbf{u}$

(iv) $\mathbf{u}+(-\mathbf{u})=-\mathbf{u}+\mathbf{u}=\mathbf{0}$

(v) $c(\mathbf{u}+\mathbf{v})=c\mathbf{u}+c\mathbf{v}$

(vi) $(c+d)\mathbf{u}=c\mathbf{u}+d\mathbf{u}$

(vii) $c(d\mathbf{u})=d(c\mathbf{u})=(cd)\mathbf{u}$

(viii) $1\mathbf{u}=\mathbf{u}$

[1] The term $-\mathbf{u}$ denotes $(-1)\mathbf{u}$.

Definition: Linear Combination

A linear combination of vectors $\mathbf{v}_1, \dots \mathbf{v}_p$ in $\mathbb{R}^n$ is a vector $\mathbf{y}$ of the form

$$\mathbf{y} = c_1\mathbf{v}_1 + \cdots + c_p\mathbf{v}_p$$

where the scalars $c_1, \dots, c_p$ are often called weights.

For example, suppose we have two vectors $\mathbf{v}_1$ and $\mathbf{v}_2$. Some linear combinations include

$$3\mathbf{v}_1, \quad -2\mathbf{v}_1 + 4\mathbf{v}_2, \quad \frac{1}{3}\mathbf{v}_2 + \sqrt{2}\mathbf{v}_1, \quad \text{and} \quad \mathbf{0} = 0\mathbf{v}_1 + 0\mathbf{v}_2.$$

Example

Let $\mathbf{a}_1 = \begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix}$, $\mathbf{a}_2 = \begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix}$, and $\mathbf{b} = \begin{bmatrix} -2 \\ -2 \\ -3 \end{bmatrix}$. Determine if $\mathbf{b}$ can be written as a linear combination of $\mathbf{a}_1$ and $\mathbf{a}_2$.

Do there exist scalers x_1 and x_2 such that $x_1\vec{a}_1 + x_2\vec{a}_2 = \vec{b}$?

$$x_1 \begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix} + x_2 \begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} -2 \\ -2 \\ -3 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ -2x_1 \\ -x_1 \end{bmatrix} + \begin{bmatrix} 3x_2 \\ 0 \\ 2x_2 \end{bmatrix} = \begin{bmatrix} -2 \\ -2 \\ -3 \end{bmatrix} \Rightarrow \begin{bmatrix} x_1 + 3x_2 \\ -2x_1 \\ -x_1 + 2x_2 \end{bmatrix} = \begin{bmatrix} -2 \\ -2 \\ -3 \end{bmatrix}$$

This gives the linear system

$x_1 + 3x_2 = -2$
$-2x_1 \qquad = -2$
$-x_1 + 2x_2 = -3$

whose augmented matrix is

$$\begin{array}{ccc} \begin{bmatrix} 1 & 3 & -2 \\ -2 & 0 & -2 \\ -1 & 2 & -3 \end{bmatrix} \\ \vec{a}_1 \quad \vec{a}_2 \quad \vec{b} \end{array}$$

$2R_1 + R_2 \to R_2, \quad R_1 + R_3 \to R_3$

$$\begin{bmatrix} 1 & 3 & -2 \\ 0 & 6 & -6 \\ 0 & 5 & -5 \end{bmatrix}$$

$\frac{1}{6}R_2 \to R_2$ then

$-5R_2 + R_3 \to R_3$

$$\begin{bmatrix} 1 & 3 & -2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

The last column is not a pivot column, so the system is consistent - i.e. $\vec{b}$ is a linear combination of $\vec{a}_1, \vec{a}_2$.

Going to rref $\quad -3R_2 + R_1 \to R_1$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

This shows that the weights are $x_1 = 1$ and $x_2 = -1$.

Some Convenient Notation

Letting $\mathbf{a}_1 = \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix}$, $\mathbf{a}_2 = \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix}$, and in general $\mathbf{a}_j = \begin{bmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \end{bmatrix}$, for $j = 1, ..., n$, we can denote the $m \times n$ matrix whose columns are these vectors by

$$[\mathbf{a}_1 \quad \mathbf{a}_2 \quad \cdots \quad \mathbf{a}_n] = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}.$$

Note that each vector $\mathbf{a}_j$ is a vector in $\mathbb{R}^m$.

Vector and Matrix Equations

The vector equation

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n = \mathbf{b}$$

has the same solution set as the linear system whose augmented matrix is

$$\begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n & \mathbf{b} \end{bmatrix}. \tag{1}$$

In particular, $\mathbf{b}$ is a linear combination of the vectors $\mathbf{a}_1, \ldots, \mathbf{a}_n$ if and only if the linear system whose augmented matrix is given in (1) is consistent.

Definition of **Span**

Let $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ be a set of vectors in $\mathbb{R}^n$. The set of all linear combinations of $\mathbf{v}_1, \ldots, \mathbf{v}_p$ is denoted by

$$\text{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_p\} = \text{Span}(S).$$

It is called the **subset of $\mathbb{R}^n$ spanned by (a.k.a. generated by)** the set $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$.

To say that a vector $\mathbf{b}$ is in $\text{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ means that there exists a set of scalars $c_1, \ldots, c_p$ such that $\mathbf{b}$ can be written as

$$c_1\mathbf{v}_1 + \cdots + c_p\mathbf{v}_p.$$

Span

If $\mathbf{b}$ is in Span$\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$, then $\mathbf{b} = c_1\mathbf{v}_1 + \cdots + c_p\mathbf{v}_p$. From the previous result, we know this is equivalent to saying that the vector equation

$$x_1\mathbf{v}_1 + \cdots + x_p\mathbf{v}_p = \mathbf{b}$$

has a solution. This is in turn the same thing as saying the linear system with augmented matrix $[\mathbf{v}_1 \ \cdots \ \mathbf{v}_p \ \mathbf{b}]$ is consistent.

Examples

Let $\mathbf{a}_1 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$, and $\mathbf{a}_2 = \begin{bmatrix} -1 \\ 4 \\ -2 \end{bmatrix}$.

(a) Determine if $\mathbf{b} = \begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix}$ is in Span$\{\mathbf{a}_1, \mathbf{a}_2\}$.

Yes if the system with augment matrix $[\vec{a}_1, \vec{a}_2\ b]$ is consistent, and NO if its not.

$$\begin{bmatrix} 1 & -1 & 4 \\ 1 & 4 & 2 \\ 2 & -2 & 1 \end{bmatrix} \quad \begin{matrix} -R_1 + R_2 \to R_2 \\ -2R_1 + R_3 \to R_3 \end{matrix}$$

$$\begin{bmatrix} 1 & -1 & 4 \\ 0 & 5 & -2 \\ 0 & 0 & -7 \end{bmatrix}$$

The last column's a pivot column so the system is inconsistent.

$\vec{b}$ is not in Span $\{\vec{a}_1, \vec{a}_2\}$

(b) Determine if $\mathbf{b} = \begin{bmatrix} 5 \\ -5 \\ 10 \end{bmatrix}$ is in Span$\{\mathbf{a}_1, \mathbf{a}_2\}$.

$$\begin{bmatrix} 1 & -1 & 5 \\ 1 & 4 & -5 \\ 2 & -2 & 10 \end{bmatrix}$$

$-R_1 + R_2 \to R_2$

$-2R_1 + R_3 \to R_3$

$$\begin{bmatrix} 1 & -1 & 5 \\ 0 & 5 & -10 \\ 0 & 0 & 0 \end{bmatrix}$$

Yes the system is consistent, $\vec{b}$ is in Span$\{\vec{a}_1, \vec{a}_2\}$

$\frac{1}{5} R_2 \to R_2$

then $\quad R_2 + R_1 \to R_1$

$$\begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}$$

So $\quad \vec{b} = 3\vec{a}_1 + (-2)\vec{a}_2$

Another Example

Give a geometric description of the subset of $\mathbb{R}^2$ given by $\text{Span}\left\{\begin{bmatrix}1\\0\end{bmatrix}\right\}$.

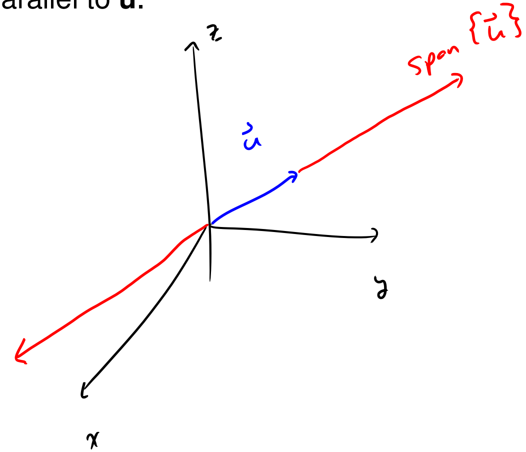
If $\vec{x}$ is in $\text{Span}\left\{\begin{bmatrix}1\\0\end{bmatrix}\right\}$ then

$\vec{x} = x_1\begin{bmatrix}1\\0\end{bmatrix}$ for some scalar x_1

this is $(x_1, 0)$, the x axis in $\mathbb{R}^2$

Span{**u**} in $\mathbb{R}^3$

If **u** is any nonzero vector in $\mathbb{R}^3$, then Span$\{\mathbf{u}\}$ is a line through the origin parallel to **u**.



Span$\{\mathbf{u},\mathbf{v}\}$ in $\mathbb{R}^3$

If $\mathbf{u}$ and $\mathbf{v}$ are nonzero, and nonparallel vectors in $\mathbb{R}^3$, then Span$\{\mathbf{u},\mathbf{v}\}$ is a plane containing the origin parallel to both vectors.

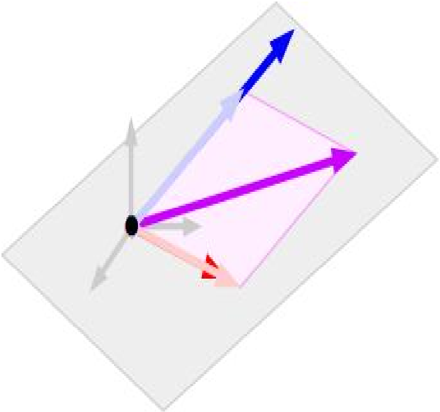


Figure: The red and blue vectors are **u** and **v**. The plane is the collection of all possible linear combinations. (A purple representative is shown.)

Example

Let $\mathbf{u} = (1, 1)$ and $\mathbf{v} = (0, 2)$ in $\mathbb{R}^2$. Show that for every pair of real numbers a and b, that (a, b) is in $\text{Span}\{\mathbf{u}, \mathbf{v}\}$.

We can use the augmented matrix $\left[\vec{u} \;\; \vec{v} \;\; \begin{bmatrix} a \\ b \end{bmatrix}\right]$

$$\begin{bmatrix} 1 & 0 & a \\ 1 & 2 & b \end{bmatrix} \quad -R_1 + R_2 \to R_2$$

$$\begin{bmatrix} 1 & 0 & a \\ 0 & 2 & b-a \end{bmatrix}$$

This shows consistency since the last column is not a pivot column.

So (a,b) is in $\text{span}\{\vec{u},\vec{v}\}$ for all pairs (a,b).

In fact $\begin{bmatrix} a \\ b \end{bmatrix} = a\vec{u} + \frac{b-a}{2}\vec{v}$

This tells us that $\text{span}\{\vec{u},\vec{v}\}$ is $\mathbb{R}^2$.

Section 1.4: The Matrix Equation $A\mathbf{x} = \mathbf{b}$.

Definition Let A be an $m \times n$ matrix whose columns are the vectors $\mathbf{a}_1, \mathbf{a}_2, \cdots, \mathbf{a}_n$ (each in $\mathbb{R}^m$), and let $\mathbf{x}$ be a vector in $\mathbb{R}^n$. Then the product of A and $\mathbf{x}$, denoted by

$$A\mathbf{x}$$

is the linear combination of the columns of A whose weights are the corresponding entries in $\mathbf{x}$. That is

$$A\mathbf{x} = x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n.$$

(Note that the result is a vector in $\mathbb{R}^m$!)

Example

Find the product $A\mathbf{x}$. Simplify to the extent possible.

$$A = \begin{bmatrix} 1 & 0 & -3 \\ -2 & -1 & 4 \end{bmatrix} \quad \mathbf{x} = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$$

$$A\vec{x} = 2\begin{bmatrix} 1 \\ -2 \end{bmatrix} + 1\begin{bmatrix} 0 \\ -1 \end{bmatrix} + (-1)\begin{bmatrix} -3 \\ 4 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ -4 \end{bmatrix} + \begin{bmatrix} 0 \\ -1 \end{bmatrix} + \begin{bmatrix} 3 \\ -4 \end{bmatrix} = \begin{bmatrix} 2+0+3 \\ -4-1-4 \end{bmatrix} = \begin{bmatrix} 5 \\ 9 \end{bmatrix}$$

Example

Find the product $A\mathbf{x}$. Simplify to the extent possible.

$$A = \begin{bmatrix} 2 & 4 \\ -1 & 1 \\ 0 & 3 \end{bmatrix} \quad \mathbf{x} = \begin{bmatrix} -3 \\ 2 \end{bmatrix}$$

$$A\vec{x} = -3\begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} + 2\begin{bmatrix} 4 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} -6 \\ 3 \\ 0 \end{bmatrix} + \begin{bmatrix} 8 \\ 2 \\ 6 \end{bmatrix}$$

$$= \begin{bmatrix} -6+8 \\ 3+2 \\ 0+6 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ 6 \end{bmatrix}$$

Example

Write the linear system as a vector equation and then as a matrix equation of the form $A\mathbf{x} = \mathbf{b}$.

$$\begin{array}{rcrcrcr} 2x_1 & - & 3x_2 & + & x_3 & = & 2 \\ x_1 & + & x_2 & + & & = & -1 \end{array}$$

Vector eqn: $x_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} -3 \\ 1 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$

Matrix eqn: $\begin{bmatrix} 2 & -3 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$

Theorem

If A is the $m \times n$ matrix whose columns are the vectors $\mathbf{a}_1, \mathbf{a}_2, \cdots, \mathbf{a}_n$, and $\mathbf{b}$ is in $\mathbb{R}^m$, then the matrix equation

$$A\mathbf{x} = \mathbf{b}$$

has the same solution set as the vector equation

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n = \mathbf{b}$$

which, in turn, has the same solution set as the linear system of equations whose augmented matrix is

$$[\mathbf{a}_1 \quad \mathbf{a}_2 \quad \cdots \quad \mathbf{a}_n \quad \mathbf{b}].$$

Corollary

The equation $A\mathbf{x} = \mathbf{b}$ has a solution if and only if $\mathbf{b}$ is a linear combination of the columns of A.

In other words, the corresponding linear system is consistent if and only if $\mathbf{b}$ is in $\text{Span}\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n\}$.

Example

Characterize the set of all vectors $\mathbf{b} = (b_1, b_2, b_3)$ such that $A\mathbf{x} = \mathbf{b}$ has a solution where

$$A = \begin{bmatrix} 1 & 3 & 4 \\ -4 & 2 & -6 \\ -3 & -2 & -7 \end{bmatrix}.$$

$$\begin{bmatrix} 1 & 3 & 4 & b_1 \\ -4 & 2 & -6 & b_2 \\ -3 & -2 & -7 & b_3 \end{bmatrix}$$

$4R_1 + R_2 \to R_2$

$3R_1 + R_3 \to R_3$

$$\begin{matrix} 4 & 12 & 16 & 4b_1 \\ -4 & 2 & -6 & b_2 \end{matrix}$$

$$\begin{matrix} 3 & 9 & 12 & 3b_1 \\ -3 & -2 & -7 & b_3 \end{matrix}$$

$$\begin{bmatrix} 1 & 3 & 4 & b_1 \\ 0 & 14 & 10 & 4b_1+b_2 \\ 0 & 7 & 5 & 3b_1+b_3 \end{bmatrix}$$

$R_2 \leftrightarrow R_3$

$$\begin{bmatrix} 1 & 3 & 4 & b_1 \\ 0 & 7 & 5 & 3b_1+b_3 \\ 0 & 14 & 10 & 4b_1+b_2 \end{bmatrix}$$

$-2R_2 + R_3 \to R_3$

$$\begin{matrix} 0 & -14 & -10 & -6b_1-2b_3 \\ 0 & 14 & 10 & 4b_1+b_2 \end{matrix}$$

$$\begin{bmatrix} 1 & 3 & 4 & b_1 \\ 0 & 7 & 5 & 3b_1+b_3 \\ 0 & 0 & 0 & -2b_1+b_2-2b_3 \end{bmatrix}$$

This is consistent only if

$$-2b_1+b_2-2b_3 = 0$$

This requires $b_1 = \frac{1}{2} b_2 - b_3$

So all such $\vec{b}$ vectors have the form

$$\vec{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{2}b_2 - b_3 \\ b_2 \\ b_3 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{2}b_2 \\ b_2 \\ 0 \end{bmatrix} + \begin{bmatrix} -b_3 \\ 0 \\ b_3 \end{bmatrix}$$

$$\vec{b} = \frac{1}{2}b_2 \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + b_3 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

All such $\vec{b}$'s belong to

$$\text{Span}\left\{\begin{bmatrix}1\\2\\0\end{bmatrix}, \begin{bmatrix}-1\\0\\1\end{bmatrix}\right\}$$

Theorem (first in a string of equivalency theorems)

Let A be an $m \times n$ matrix. Then the following are logically equivalent (i.e. they are either all true or are all false).

(a) For each $\mathbf{b}$ in $\mathbb{R}^m$, the equation $A\mathbf{x} = \mathbf{b}$ has a solution.

(b) Each $\mathbf{b}$ in $\mathbb{R}^m$ is a linear combination of the columns of A.

(c) The columns of A span $\mathbb{R}^m$.

(d) A has a pivot position in every row.

(Note that statement (d) is about the *coefficient* matrix A, not about an augmented matrix $[A \quad \mathbf{b}]$.)