# August 24 Math 2306 sec 51 Fall 2015

## Section 1.2: Initial Value Problems

An initial value problem consists of an ODE with additional conditions.

Solve the equation [1]

$$\frac{d^n y}{dx^n} = f(x, y, y', \dots, y^{(n-1)}) \tag{1}$$

subject to the *initial conditions*

$$y(x_0) = y_0, \quad y'(x_0) = y_1, \quad \dots, y^{(n-1)}(x_0) = y_{n-1}. \tag{2}$$

The problem (1)–(2) is called an *initial value problem* (IVP).

[1] on some interval $I$ containing $x_0$.

# Example

**Part 1**

Show that for any constant $c$ the relation $x^2 + y^2 = c$ is an implicit solution of the ODE

$$\frac{dy}{dx} = -\frac{x}{y}$$

Suppose $x^2+y^2=c$ and find $\frac{dy}{dx}$ using implicit differentiation.

$$2x + 2y\frac{dy}{dx} = 0$$

$$\Rightarrow \quad x + y\frac{dy}{dx} = 0$$

$$\Rightarrow \quad y\frac{dy}{dx} = -x \quad \Rightarrow \quad \frac{dy}{dx} = \frac{-x}{y} \quad (\text{for } y \neq 0)$$

Hence if $y$ satisfies $x^2+y^2=C$ it also satisfies $\dfrac{dy}{dx} = \dfrac{-x}{y}$.

# Example

**Part 2**

Use the preceding results to find an explicit solution of the IVP

$$\frac{dy}{dx} = -\frac{x}{y}, \quad y(0) = -2$$

means $y=-2$ when $x=0$

From part 1 we know that solution(s) satisfy $x^2+y^2=C$ for constant $C$.

Impose $y(0)=-2$ $\quad 0^2+(-2)^2 = C \Rightarrow C=4$

so $x^2 + y^2 = 4$ solve for $y$

$y^2 = 4 - x^2 \Rightarrow y = \sqrt{4 - x^2}$ or $y = -\sqrt{4-x^2}$

gives $y(0) = 2$

gives $y(0) = -2$

Only the latter has the correct sign.

The explicit solution is $y = -\sqrt{4-x^2}$

# Graphical Interpretation

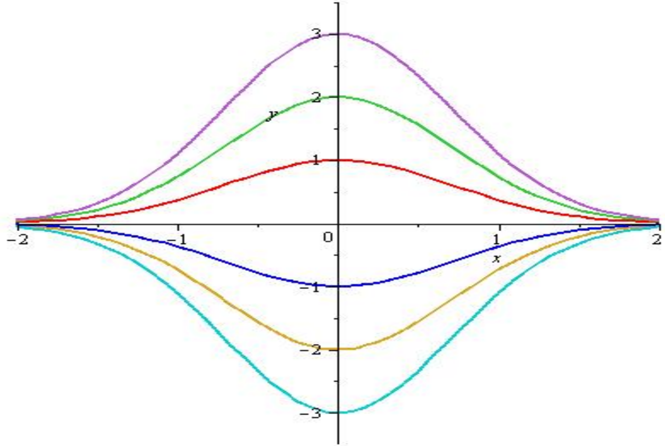


Figure: Each curve solves $y' + 2xy = 0$, $y(0) = y_0$. Each colored curve corresponds to a different value of $y_0$

# Example

$x = c_1 \cos(2t) + c_2 \sin(2t)$ is a 2-parameter family of solutions of the ODE $x'' + 4x = 0$. Find a solution of the IVP

$$x'' + 4x = 0, \quad x\left(\frac{\pi}{2}\right) = -1, \quad x'\left(\frac{\pi}{2}\right) = 4$$

$$x(t) = c_1 \cos(2t) + c_2 \sin(2t)$$

$$x\left(\frac{\pi}{2}\right) = c_1 \cos\left(2 \cdot \frac{\pi}{2}\right) + c_2 \sin\left(2 \cdot \frac{\pi}{2}\right) = -1$$

$$c_1(-1) + c_2(0) = -1 \Rightarrow c_1 = 1$$

$$x'(t) = -2c_1 \sin(2t) + 2c_2 \cos(2t)$$

$$x'\left(\frac{\pi}{2}\right) = -2c_1 \sin\left(2\cdot\frac{\pi}{2}\right) + 2c_2 \cos\left(2\cdot\frac{\pi}{2}\right) = 4$$

$$-2c_1(0) + 2c_2(-1) = 4 \implies c_2 = -2$$

So the solution to the IVP is

$$x(t) = \cos(2t) - 2\sin(2t)$$

$c_1 = 1$ and $c_2 = -2$

# Existence and Uniqueness

Two important questions we can always pose (and sometimes answer) are

(1) Does an IVP have a solution? (existence) and

(2) If it does, is there just one? (uniqueness)

Hopefully it's obvious that we can't solve $\underbrace{\left(\frac{dy}{dx}\right)^2 + 1}_{\text{always } \geq 1} = \underbrace{-y^2}_{\text{always } \leq 0}.$

# Uniqueness

Consider the IVP

$$\frac{dy}{dx} = x\sqrt{y} \quad y(0) = 0$$

Verify that $y = \frac{x^4}{16}$ is a solution of the IVP. And find a second solution of the IVP by clever guessing.

Verify $y = \frac{x^4}{16}$ solves the IVP:

ODE: $\frac{dy}{dx} = \frac{4x^3}{16} = \frac{x^3}{4}$ , $x\sqrt{y} = x\sqrt{\frac{x^4}{16}} = x\left|\frac{x^2}{4}\right|$

$= x\left(\frac{x^2}{4}\right) = \frac{x^3}{4}$

$\frac{x^3}{4} = \frac{x^3}{4}$ ✓

Initial Condition: $y(0) = \frac{0^4}{16} = 0$ as required.

$y = \frac{x^4}{16}$ solves the ODE and the initial condition, hence it solves the IVP.

$$\frac{dy}{dx} = x\sqrt{y}, \quad y(0) = 0$$

The constant solution $y=0$ also solves the IVP. Note $\frac{d}{dx}0 = 0 = x\sqrt{0}$ and

$$\underset{y}{0} = 0 \text{ when } x = 0.$$

# Section 2.2: Separation of Variables

The simplest type of equation we could encounter would be of the form

$$\frac{dy}{dx} = g(x).$$

For example, solve the ODE

$$\frac{dy}{dx} = 4e^{2x} + 1.$$

$$\frac{dy}{dx}\,dx = \left(4e^{2x}+1\right)dx$$

$$\int \frac{dy}{dx}\,dx = \int \left(4e^{2x}+1\right)dx$$

$$\boxed{y = 2e^{2x} + x + C}$$

a one parameter family of solutions.

Recall

$$\frac{dy}{dx}\,dx = dy$$

# Separable Equations

**Definition:** The first order equation $y' = f(x, y)$ is said to be **separable** if the right side has the form

$$f(x, y) = g(x)h(y).$$

That is, a separable equation is one that has the form

$$\frac{dy}{dx} = g(x)h(y).$$

function of x-alone

times

function of y-alone

Determine which (if any) of the following are separable.

(a) $\dfrac{dy}{dx} = x^3 y$

This is separable with

$g(x) = x^3$ and $h(y) = y$

(b) $\dfrac{dy}{dx} = 2x + y$

Not separable It can't be written as function of x times function of y

$2x + y = x\left(2 + \frac{y}{x}\right) = y\left(\frac{2x}{y} + 1\right)$

(c) $\dfrac{dy}{dx} = \sin(xy^2)$

Not separable

(d) $\dfrac{dy}{dt} - te^{t-y} = 0$

This is separable with

$g(t) = te^t$ and

$h(y) = e^{-y}$

$\dfrac{dy}{dt} = te^{t-y} = te^t e^{-y}$

# Solving Separable Equations

Recall that from $\frac{dy}{dx} = g(x)$, we can integrate both sides

$$\int \frac{dy}{dx}\,dx = \int g(x)\,dx.$$

$y = G(x) + C$

where $G(x)$ is any

anti derivative of $g(x)$

We'll use this observation!

# Solving Separable Equations

Let's assume that it's safe to divide by $h(y)$ and let's set $p(y) = 1/h(y)$. We solve (usually find an implicit solution) by **separating the variables.**

$$\frac{dy}{dx} = g(x)h(y) \qquad \Longrightarrow \frac{1}{h(y)}\frac{dy}{dx} = g(x)$$

$$\Rightarrow \quad p(y)\,\frac{dy}{dx}\,dx = g(x)\,dx$$

$$p(y)\,dy = g(x)\,dx$$

$$\int p(y)\,dy = \int g(x)\,dx$$

$$P(y) = G(x) + C$$

where $P$ is any antiderivative of $p(y)$

and $G$ is any antiderivative of $g(x)$

$P(y) = G(x) + C$ defines a family of implicit solutions of $\frac{dy}{dx} = g(x)h(y)$