August 24 Math 2306 sec. 53 Fall 2018

Section 4: First Order Equations: Linear

A first order linear equation has the form

$$a_1(x)\frac{dy}{dx} + a_0(x)y = g(x).$$

If $g(x) = 0$ the equation is called **homogeneous.** Otherwise it is called **nonhomogeneous.**

Provided $a_1(x) \neq 0$ on the interval I of definition of a solution, we can write the **standard form** of the equation

$P(x) = \frac{a_0(x)}{a_1(x)}$

$$\frac{dy}{dx} + P(x)y = f(x).$$

$f(x) = \frac{g(x)}{a_1(x)}$

We'll be interested in equations (and intervals I) for which P and f are continuous on I.

Solutions (the General Solution)

$$\frac{dy}{dx} + P(x)y = f(x).$$

It turns out the solution will always have a basic form of $y = y_c + y_p$ where

- y_c is called the **complementary** solution and would solve the equation
$$\frac{dy}{dx} + P(x)y = 0$$
(called the associated homogeneous equation), and
- y_p is called the **particular** solution, and is heavily influenced by the function $f(x)$.

The cool thing is that our solution method will get both parts in one process—we won't get this benefit with higher order equations!

Motivating Example

$$x^2 \frac{dy}{dx} + 2xy = e^x$$

This is not in standard form. We could put it in standard form, but we won't.

Note that the left side

$$x^2 \frac{dy}{dx} + 2xy = x^2 \left(\frac{dy}{dx}\right) + \left(\frac{d}{dx} x^2\right) y$$

$$= \frac{d}{dx}\left(x^2 y\right)$$

So the equation is

$$\frac{d}{dx}\left(x^2 y\right) = e^x$$

Goal: Solve for y

Integrate $$\int \frac{d}{dx}\left(x^2 y\right) dx = \int e^x \, dx$$

$$x^2 y = e^x + C$$

Divide :

$$y = \frac{e^x + C}{x^2} = \frac{e^x}{x^2} + \frac{C}{x^2}$$

Derivation of Solution via Integrating Factor

Solve the equation in standard form

$$\frac{dy}{dx} + P(x)y = f(x)$$

The approach is to make the left side into a product rule. We'll multiply both sides by some function $\mu(x)$ to get this product rule.

We'll assume $\mu(x) > 0$.

$$\mu \frac{dy}{dx} + \mu P(x) y = \mu f(x)$$

We want the left to be $\frac{d}{dx}[\mu y]$.

$$\frac{d}{dx}[\mu y] = \mu\frac{dy}{dx} + \frac{d\mu}{dx}y = \mu\frac{dy}{dx} + \mu P(x)\, y$$

need these to match

We get an equation for μ

$$\frac{d\mu}{dx}y = \mu P(x)\, y$$

$$\Rightarrow \quad \frac{d\mu}{dx} = \mu P(x)$$

a separable equation for μ

Solve this

$$\frac{1}{\mu}\frac{d\mu}{dx} = P(x)$$

$$\int \frac{1}{\mu}\, d\mu = \int p(x)\, dx$$

$$\ln \mu = \int p(x)\, dx$$

$$\Longrightarrow \quad \mu = e^{\int p(x)\, dx}$$

This is an integrating factor.

For this μ, the ODE is

$$\frac{d}{dx}\left[\mu y\right] = \mu f(x)$$

Integrate and divide by μ

$$\int \frac{d}{dx}[\mu y]\, dx = \int \mu(x) f(x)\, dx$$

$$\mu y = \int \mu(x) f(x)\, dx + C$$

$$\boxed{y = \mu^{-1} \int \mu(x) f(x)\, dx + C\mu^{-1}}$$

General Solution of First Order Linear ODE

- Put the equation in standard form $y' + P(x)y = f(x)$, and correctly identify the function $P(x)$.
- Obtain the integrating factor $\mu(x) = \exp\left(\int P(x)\,dx\right)$.
- Multiply both sides of the equation (in standard form) by the integrating factor μ. The left hand side **will always** collapse into the derivative of a product

$$\frac{d}{dx}[\mu(x)y] = \mu(x)f(x).$$

- Integrate both sides, and solve for y.

$$y(x) = \frac{1}{\mu(x)}\int \mu(x)f(x)\,dx = e^{-\int P(x)\,dx}\left(\int e^{\int P(x)\,dx}f(x)\,dx + C\right)$$

Solve the ODE

Already in standard form

$$\frac{dy}{dx} + y = 3xe^{-x}$$

$P(x) = 1$

$\mu = e^{\int P(x)\,dx} = e^{\int 1\,dx} = e^{x}$

Mult by μ $\quad e^{x}\frac{dy}{dx} + e^{x}y = e^{x}(3xe^{-x})$

$\frac{d}{dx}\left[e^{x}y\right] = 3x$ $\qquad e^{x}\cdot e^{-x} = 1$

$$\int \frac{d}{dx}\left[e^x y\right] dx = \int 3x \, dx$$

$$e^x y = \frac{3}{2}x^2 + C$$

$$\boxed{y = \underbrace{\frac{3}{2}x^2 e^{-x}}_{y_p} + \underbrace{Ce^{-x}}_{y_c}}$$