August 24 Math 2306 sec 54 Fall 2015

Section 1.2: Initial Value Problems

An initial value problem consists of an ODE with additional conditions.

Solve the equation [1]

$$\frac{d^n y}{dx^n} = f(x, y, y', \dots, y^{(n-1)}) \tag{1}$$

subject to the *initial conditions*

$$y(x_0) = y_0, \quad y'(x_0) = y_1, \quad \dots, y^{(n-1)}(x_0) = y_{n-1}. \tag{2}$$

The problem (1)–(2) is called an *initial value problem* (IVP).

[1] on some interval I containing x_0.

Example

Given that $y = c_1 x + \frac{c_2}{x}$ is a 2-parameter family of solutions of $x^2 y'' + xy' - y = 0$, solve the IVP

$$x^2 y'' + xy' - y = 0, \quad y(1) = 1, \quad y'(1) = 3$$

All solutions to the ODE are of the form

$$y = c_1 x + \frac{c_2}{x}.$$

Now impose $y = 1$ when $x = 1$ and $y' = 3$ when $x = 1$

$$y' = c_1 - \frac{c_2}{x^2}$$

$$y(1) = c_1(1) + \frac{c_2}{1} = 1 \implies c_1 + c_2 = 1$$

$$y'(1) = c_1 - \frac{c_2}{1^2} = 3 \implies c_1 - c_2 = 3$$

add eqns

$$2c_1 = 4 \implies c_1 = 2$$

subtract eqns

$$2c_2 = -2 \implies c_2 = -1$$

So to solve the ODE and the conditions $y(1)=1$, $y'(1)=3$ the $c_1 = 2$ and $c_2 = -1$.

The solution to the IVP is $\boxed{y = 2x - \frac{1}{x}}$

Example

Part 1

Show that for any constant c the relation $x^2 + y^2 = c$ is an implicit solution of the ODE

$$\frac{dy}{dx} = -\frac{x}{y}$$

We'll use implicit differentiation to show that if y satisfies $x^2+y^2=c$, then it satisfies

$$\frac{dy}{dx} = \frac{-x}{y}$$

$$x^2+y^2 = c \quad \Rightarrow \quad 2x + 2y\frac{dy}{dx} = 0$$

$$\Rightarrow \quad x + y\frac{dy}{dx} = 0 \quad \Rightarrow \quad y\frac{dy}{dx} = -x$$

$$\frac{dy}{dx} = \frac{-x}{y}$$

So $x^2+y^2=C$ defines an implicit soln. to the ODE for any C $(C>0)$.

Example

Part 2

Use the preceding results to find an explicit solution of the IVP

$$\frac{dy}{dx} = -\frac{x}{y}, \quad y(0) = -2$$

Solutions to the ODE are defined implicitly by

$$x^2 + y^2 = C.$$

Impose the condition $x=0$ and $y=-2$

$$0^2 + (-2)^2 = C \quad \Rightarrow \quad C=4$$

$$x^2+y^2=4 \implies y^2=4-x^2$$

so $y=\sqrt{4-x^2}$ or $y=-\sqrt{4-x^2}$

this gives $y(0)=2$

not the soln. to the IVP

this gives $y(0)=-2$

The explicit solution to the IVP is

$$y=-\sqrt{4-x^2}\,.$$

Graphical Interpretation

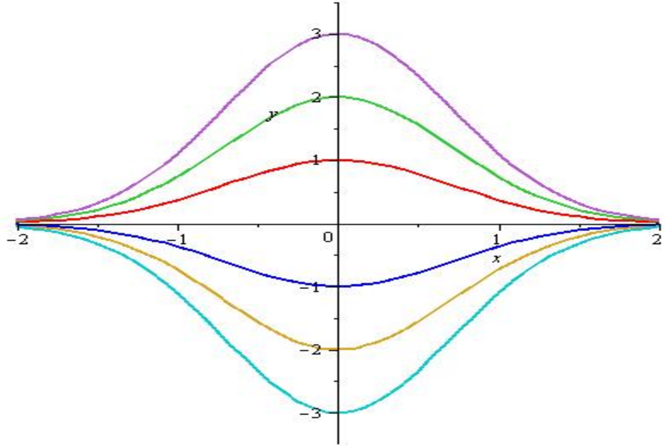

Figure: Each curve solves $y' + 2xy = 0$, $y(0) = y_0$. Each colored curve corresponds to a different value of y_0

Example

$x = c_1\cos(2t) + c_2\sin(2t)$ is a 2-parameter family of solutions of the ODE $x'' + 4x = 0$. Find a solution of the IVP

$$x'' + 4x = 0, \quad x\left(\frac{\pi}{2}\right) = -1, \quad x'\left(\frac{\pi}{2}\right) = 4$$

Solutions to the ODE are

$$X = C_1 \cos(2t) + C_2 \sin(2t)$$

$$X' = -2C_1 \sin(2t) + 2C_2 \cos(2t)$$

$$x\left(\frac{\pi}{2}\right) = C_1 \cos\left(2\cdot\frac{\pi}{2}\right) + C_2 \sin\left(2\cdot\frac{\pi}{2}\right) = -1$$

$$c_1(-1) + c_2(0) = -1 \implies c_1 = 1$$

$$x'\left(\frac{\pi}{2}\right) = -2c_1 \sin\left(2\cdot\frac{\pi}{2}\right) + 2c_2 \cos\left(2\cdot\frac{\pi}{2}\right) = 4$$

$$-2c_1(0) + 2c_2(-1) = 4 \implies c_2 = -2$$

The solution to the IVP is

$$x = \cos(2t) - 2\sin(2t).$$

Existence and Uniqueness

Two important questions we can always pose (and sometimes answer) are

(1) Does an IVP have a solution? (existence) and

(2) If it does, is there just one? (uniqueness)

Hopefully it's obvious that we can't solve $\underbrace{\left(\frac{dy}{dx}\right)^2+1}_{\text{always} \geq 1} = \underbrace{-y^2}_{\text{always} \leq 0}.$

Uniqueness

Consider the IVP

$$\frac{dy}{dx} = x\sqrt{y} \quad y(0) = 0$$

Verify that $y = \frac{x^4}{16}$ is a solution of the IVP. And find a second solution of the IVP by clever guessing.

We want to show that $y = \frac{x^4}{16}$ solves the ODE and the initial condition.

ODE: $y = \frac{x^4}{16} \Rightarrow \frac{dy}{dx} = \frac{4x^3}{16} = \frac{x^3}{4}$

$$x\sqrt{y} = x\sqrt{\frac{x^4}{16}} = x\left|\frac{x^2}{4}\right| = x\left(\frac{x^2}{4}\right) = \frac{x^3}{4}$$

$$\frac{dy}{dx} = \frac{x^3}{4} = x\sqrt{y}$$

Initial condition: $y(0) = \frac{0^4}{16} = 0$ i.e. $y(0)=0$

So $y = \frac{x^4}{16}$ is a solution to the IVP.

$$\frac{dy}{dx} = x\sqrt{y}\,, \quad y(0)=0$$

If we guess a constant solution we get

$$y=0$$

note $y=0 \Rightarrow \frac{dy}{dx} = 0 = x\sqrt{0}$ It solves the ODE

A second solution is the trivial solution $y=0$.

Section 2.2: Separation of Variables

The simplest type of equation we could encounter would be of the form

$$\frac{dy}{dx} = g(x).$$

For example, solve the ODE

$$\frac{dy}{dx} = 4e^{2x} + 1.$$

$$\frac{dy}{dx}\,dx = (4e^{2x}+1)\,dx$$

$$\int \frac{dy}{dx}\,dx = \int (4e^{2x}+1)\,dx$$

$$\int dy = \int (4e^{2x}+1)\,dx$$

Recall

$$dy = \frac{dy}{dx}\,dx$$

$$y = 2e^{2x} + x + C$$

Separable Equations

Definition: The first order equation $y' = f(x, y)$ is said to be **separable** if the right side has the form

$$f(x, y) = g(x)h(y).$$

That is, a separable equation is one that has the form

$$\frac{dy}{dx} = g(x)h(y).$$

function of x only

times

function of y only

Determine which (if any) of the following are separable.

(a) $\frac{dy}{dx} = x^3y$ It is separable with $g(x) = x^3$ and $h(y) = 1$

(b) $\frac{dy}{dx} = 2x+y$ This is not separable.

(c) $\dfrac{dy}{dx} = \sin(xy^2)$

This is not separable.

(d) $\dfrac{dy}{dt} - te^{t-y} = 0$

This is separable w/

$g(t) = te^t$ and

$h(y) = e^{-y}$

$\dfrac{dy}{dt} = te^{t-y} = te^t \cdot e^{-y}$