Section 4: First Order Equations: Linear

A first order linear equation has the form

$$a_1(x) \frac{dy}{dx} + a_0(x)y = g(x).$$

If $g(x) = 0$ the equation is called **homogeneous**. Otherwise it is called **nonhomogeneous**.

Provided $a_1(x) \neq 0$ on the interval $I$ of definition of a solution, we can write the **standard form** of the equation

$$\frac{dy}{dx} + P(x)y = f(x).$$

We’ll be interested in equations (and intervals $I$) for which $P$ and $f$ are continuous on $I$. 
Solutions (the General Solution)

\[
\frac{dy}{dx} + P(x)y = f(x).
\]

It turns out the solution will always have a basic form of \( y = y_c + y_p \) where

- \( y_c \) is called the **complementary** solution and would solve the problem
  \[
  \frac{dy}{dx} + P(x)y = 0
  \]
  (called the associated homogeneous equation), and
- \( y_p \) is called the **particular** solution, and is heavily influenced by the function \( f(x) \).
Motivating Example

\[ x^2 \frac{dy}{dx} + 2xy = e^x \]

We solved this equation by recognizing the left side as the derivative of a product \( \frac{d}{dx} [x^2 y] \), and integrating. We found

\[ y = \frac{C + e^x}{x^2} = \frac{C}{x^2} + \frac{e^x}{x^2}. \]
Derivation of Solution via Integrating Factor

Solve the equation in standard form

\[ \frac{dy}{dx} + P(x)y = f(x) \]
General Solution of First Order Linear ODE

- Put the equation in standard form $y' + P(x)y = f(x)$, and correctly identify the function $P(x)$.
- Obtain the integrating factor $\mu(x) = \exp\left(\int P(x) \, dx\right)$.
- Multiply both sides of the equation (in standard form) by the integrating factor $\mu$. The left hand side will always collapse into the derivative of a product
  \[
  \frac{d}{dx} [\mu(x)y] = \mu(x)f(x).
  \]
- Integrate both sides, and solve for $y$.
  \[
  y(x) = \frac{1}{\mu(x)} \int \mu(x)f(x) \, dx = e^{-\int P(x) \, dx} \left( \int e^{\int P(x) \, dx} f(x) \, dx + C \right)
  \]
Solve the ODE

\[
\frac{dy}{dx} + y = 3xe^{-x}
\]
Solve the IVP

\[ x \frac{dy}{dx} - y = 2x^2, \quad x > 0 \quad y(1) = 5 \]
Verify

Just for giggles, let's verify that our solution $y = 2x^2 + 3x$ really does solve the differential equation we started with

$$x \frac{dy}{dx} - y = 2x^2.$$
Steady and Transient States

For some linear equations, the term $y_c$ decays as $x$ (or $t$) grows. For example

$$\frac{dy}{dx} + y = 3xe^{-x} \quad \text{has solution} \quad y = \frac{3}{2}x^2 + Ce^{-x}.$$

Here, $y_p = \frac{3}{2}x^2$ and $y_c = Ce^{-x}$.

Such a decaying complementary solution is called a **transient state**.

The corresponding particular solution is called a **steady state**.
Bernoulli Equations

Suppose $P(x)$ and $f(x)$ are continuous on some interval $(a, b)$ and $n$ is a real number different from 0 or 1 (not necessarily an integer). An equation of the form

$$\frac{dy}{dx} + P(x)y = f(x)y^n$$

is called a Bernoulli equation.

Observation: This equation has the flavor of a linear ODE, but since $n \neq 0, 1$ it is necessarily nonlinear. So our previous approach involving an integrating factor does not apply directly. Fortunately, we can use a change of variables to obtain a related linear equation.
Solving the Bernoulli Equation

\[ \frac{dy}{dx} + P(x)y = f(x)y^n \]  

(1)
Example

Solve the initial value problem $y' - y = -e^{2x} y^3$, subject to $y(0) = 1$. 
Exact Equations

We considered first order equations of the form

\[ M(x, y) \, dx + N(x, y) \, dy = 0. \]  \hspace{1cm} (2)

The left side is called a *differential form*. We will assume here that $M$ and $N$ are continuous on some (shared) region in the plane.

**Definition:** The equation (2) is called an **exact equation** on some rectangle $R$ if there exists a function $F(x, y)$ such that

\[ \frac{\partial F}{\partial x} = M(x, y) \quad \text{and} \quad \frac{\partial F}{\partial y} = N(x, y) \]

for every $(x, y)$ in $R$. 

Exact Equation Solution

If $M(x, y) \, dx + N(x, y) \, dy = 0$ happens to be exact, then it is equivalent to

$$\frac{\partial F}{\partial x} \, dx + \frac{\partial F}{\partial y} \, dy = 0$$

This implies that the function $F$ is constant on $R$ and solutions to the DE are given by the relation

$$F(x, y) = C$$
Recognizing Exactness

There is a theorem from calculus that ensures that if a function $F$ has first partials on a domain, and if those partials are continuous, then the second mixed partials are equal. That is,

$$\frac{\partial^2 F}{\partial y \partial x} = \frac{\partial^2 F}{\partial x \partial y}.$$  

If it is true that

$$\frac{\partial F}{\partial x} = M \quad \text{and} \quad \frac{\partial F}{\partial y} = N$$

this provides a condition for exactness, namely

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$
Exact Equations

**Theorem:** Let $M$ and $N$ be continuous on some rectangle $R$ in the plane. Then the equation

$$M(x, y) \, dx + N(x, y) \, dy = 0$$

is exact if and only if

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$$
Example
Show that the equation is exact and obtain a family of solutions.

\[(2xy - \sec^2 x) \, dx + (x^2 + 2y) \, dy = 0\]
Special Integrating Factors

Suppose that the equation $M \, dx + N \, dy = 0$ is not exact. Clearly our approach to exact equations would be fruitless as there is no such function $F$ to find. It may still be possible to solve the equation if we can find a way to morph it into an exact equation. As an example, consider the DE

$$(2y - 6x) \, dx + (3x - 4x^2 y^{-1}) \, dy = 0$$

Note that this equation is NOT exact. In particular

$$\frac{\partial M}{\partial y} = 2 \neq 3 - 8xy^{-1} = \frac{\partial N}{\partial x}.$$
Special Integrating Factors

But note what happens when we multiply our equation by the function $\mu(x, y) = xy^2$.

\[
xy^2(2y - 6x)\,dx + xy^2(3x - 4x^2y^{-1})\,dy = 0,
\implies

(2xy^3 - 6x^2y^2)\,dx + (3x^2y^2 - 4x^3y)\,dy = 0
\]
Special Integrating Factors

The function $\mu$ is called a *special integrating factor*. Finding one (assuming one even exists) may require ingenuity and likely a bit of luck. However, there are certain cases we can look for and perhaps use them to solve the occasional equation. A useful method is to look for $\mu$ of a certain form (usually $\mu = x^n y^m$ for some powers $n$ and $m$). We will restrict ourselves to two possible cases:

There is an integrating faction $\mu = \mu(x)$ depending only on $x$, or there is an integrating factor $\mu = \mu(y)$ depending only on $y$. 
Special Integrating Factor $\mu = \mu(x)$

Suppose that

$$M \, dx + N \, dy = 0$$

is NOT exact, but that

$$\mu M \, dx + \mu N \, dy = 0$$

IS exact where $\mu = \mu(x)$ does not depend on $y$. Then

$$\frac{\partial(\mu(x)M)}{\partial y} = \frac{\partial(\mu(x)N)}{\partial x}.$$ 

Let’s use the product rule in the right side.
Special Integrating Factor $\mu = \mu(x)$

$$\frac{\partial (\mu(x)M)}{\partial y} = \frac{\partial (\mu(x)N)}{\partial x}.$$
Special Integrating Factor

\[ M \, dx + N \, dy = 0 \] (3)

**Theorem:** If \( (\partial M/\partial y - \partial N/\partial x)/N \) is continuous and depends only on \( x \), then

\[
\mu = \exp \left( \int \frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} \, dx \right)
\]

is an special integrating factor for (3). If \( (\partial N/\partial x - \partial M/\partial y)/M \) is continuous and depends only on \( y \), then

\[
\mu = \exp \left( \int \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \, dy \right)
\]

is an special integrating factor for (3).
Example

Solve the equation $2xy \, dx + (y^2 - 3x^2) \, dy = 0$. 