August 24 Math 2306 sec. 56 Fall 2017

Section 4: First Order Equations: Linear

A first order linear equation has the form

$$a_1(x)\frac{dy}{dx} + a_0(x)y = g(x).$$

If $g(x) = 0$ the equation is called **homogeneous.** Otherwise it is called **nonhomogeneous.**

Provided $a_1(x) \neq 0$ on the interval I of definition of a solution, we can write the **standard form** of the equation

$$\frac{dy}{dx} + P(x)y = f(x).$$

We'll be interested in equations (and intervals I) for which P and f are continuous on I.

Solutions (the General Solution)

$$\frac{dy}{dx} + P(x)y = f(x).$$

It turns out the solution will always have a basic form of $y = y_c + y_p$ where

- y_c is called the **complementary** solution and would solve the problem
$$\frac{dy}{dx} + P(x)y = 0$$
(called the associated homogeneous equation), and
- y_p is called the **particular** solution, and is heavily influenced by the function $f(x)$.

Motivating Example

$$x^2\frac{dy}{dx}+2xy=e^x$$

We solved this equation by recognizing the left side as the derivative of a product $\frac{d}{dx}[x^2y]$, and integrating. We found

$$y=\frac{C+e^x}{x^2}=\frac{C}{x^2}+\frac{e^x}{x^2}.$$

y_c y_p

Derivation of Solution via Integrating Factor

Solve the equation in standard form

$$\frac{dy}{dx} + P(x)y = f(x)$$

Suppose $\mu(x)$ is some nonzero function.

$$\mu(x)\frac{dy}{dx} + P(x)\mu(x)\,y = \mu(x)f(x)$$

We want to choose μ so that the LHS is a product rule $\frac{d}{dx}[\mu y]$. We need

$$\frac{d}{dx}[\mu y] = \mu\frac{dy}{dx} + \frac{d\mu}{dx}y = \mu\frac{dy}{dx} + \mu P y$$

matches

must match!

Our condition on μ is

$$\frac{d\mu}{dx}y = \mu P y \implies \frac{d\mu}{dx} = \mu P$$

Separable!

Separate variables

$$\frac{1}{\mu}\frac{d\mu}{dx} = P(x)$$

$$\int \frac{1}{\mu}\frac{d\mu}{dx}\,dx = \int P(x)\,dx$$

$$\int \frac{1}{\mu}\,d\mu = \int P(x)\,dx$$

assuming $\mu > 0$

$$\ln\mu = \int P(x)\,dx$$

Exponentiate $\mu = e^{\int P(x)\,dx}$ called an integrating factor

Multiply $y' + Py = f$ by μ

$$\mu y' + \underbrace{\mu P}_{\mu'} y = \mu f$$

LHS is product rule

$$\frac{d}{dx}[\mu y] = \mu f$$

$$\int \frac{d}{dx}[\mu y]\,dx = \int \mu f\,dx$$

$$\mu y = \int \mu f \, dx$$

Solve for y. Writing the "$+C$" explicitly

$$y = \frac{1}{\mu}\left(\int \mu f \, dx + C\right)$$

$$y = \underbrace{e^{-\int P(x)dx}\left(\int e^{\int P(x)dx} f(x)\, dx\right)}_{y_p} + \underbrace{C e^{-\int P(x)dx}}_{y_c}$$

General Solution of First Order Linear ODE

- Put the equation in standard form $y' + P(x)y = f(x)$, and correctly identify the function $P(x)$.
- Obtain the integrating factor $\mu(x) = \exp\left(\int P(x)\,dx\right)$.
- Multiply both sides of the equation (in standard form) by the integrating factor μ. The left hand side **will always** collapse into the derivative of a product

$$\frac{d}{dx}[\mu(x)y] = \mu(x)f(x).$$

- Integrate both sides, and solve for y.

$$y(x) = \frac{1}{\mu(x)}\int \mu(x)f(x)\,dx = e^{-\int P(x)\,dx}\left(\int e^{\int P(x)\,dx} f(x)\,dx + C\right)$$

Solve the ODE

$$\frac{dy}{dx}+y=3xe^{-x}$$

Already in standard form.

Here, $P(x)=1$

$$\mu = e^{\int P(x)\,dx} = e^{\int dx} = e^{x+C} = e^{x}\cdot e^{C} = Ae^{x}$$

where $A=e^{C}$. Mult by μ

$$Ae^{x}\left(\frac{dy}{dx}+y\right) = Ae^{x}\left(3xe^{-x}\right)$$

we can cancel the constant factor A.

$$e^{x}\frac{dy}{dx}+e^{x}y = e^{x}(3)xe^{-x} = 3x$$

$$\frac{d}{dx}\left[e^x y\right] = 3x$$

$$\int \frac{d}{dx}\left[e^x y\right] dx = \int 3x \, dx$$

$$e^x y = 3\frac{x^2}{2} + C$$

$$y = \frac{\frac{3}{2}x^2 + C}{e^x}$$

$$\boxed{y = \frac{3}{2}x^2 e^{-x} + Ce^{-x}}$$

Solve the IVP

$$x\frac{dy}{dx} - y = 2x^2, \; x > 0 \quad y(1) = 5$$

The ODE is not in standard form.

In standard form

$$\frac{dy}{dx} - \frac{1}{x}y = \frac{2x^2}{x} = 2x$$

Here $P(x) = \frac{-1}{x}$

$$\mu = e^{\int P(x)\,dx} = e^{\int \frac{-1}{x}dx} = e^{-\ln x} = e^{\ln x^{-1}} = x^{-1}$$

$$\frac{1}{x}\frac{dy}{dx} - \frac{1}{x^2}y = \frac{1}{x}(2x) = 2$$

$$\frac{d}{dx}\left[\frac{1}{x}y\right] = 2$$

$$\int \frac{d}{dx}\left[\frac{1}{x}y\right]dx = \int 2\,dx$$

$$\frac{1}{x}y = 2x + C$$

Mult by x $$y = 2x^2 + Cx$$

Impose $y(1) = 5$ $$y(1) = 5 = 2(1^2) + C(1)$$

$$5 = 2 + C \implies C = 3$$

The soln to the IVP is

$$y = 2x^2 + 3x$$

Verify

Just for giggles, lets verify that our solution $y = 2x^2 + 3x$ really does solve the differential equation we started with

$$x\frac{dy}{dx} - y = 2x^2.$$

$$y = 2x^2 + 3x, \qquad \frac{dy}{dx} = 4x + 3$$

$$\begin{aligned} x\frac{dy}{dx} - y &= x(4x+3) - (2x^2 + 3x) \\ &= 4x^2 + \cancel{3x} - 2x^2 - \cancel{3x} \\ &= 4x^2 - 2x^2 \\ &= 2x^2 \quad \text{as expected!} \end{aligned}$$

Steady and Transient States

For some linear equations, the term y_c decays as x (or t) grows. For example

(Typo)

$$\frac{dy}{dx} + y = 3xe^{-x} \quad \text{has solution} \quad y = \frac{3}{2}x^2 e^{-x} + Ce^{-x}.$$

$$\text{Here,} \quad y_p = \frac{3}{2}x^2 e^{-x} \text{ and } \quad y_c = Ce^{-x}.$$

Such a decaying complementary solution is called a **transient state.**

The corresponding particular solution is called a **steady state.**