August 24 Math 2306 sec. 57 Fall 2017

Section 4: First Order Equations: Linear

A first order linear equation has the form

$$a_1(x)\frac{dy}{dx} + a_0(x)y = g(x).$$

If $g(x) = 0$ the equation is called **homogeneous.** Otherwise it is called **nonhomogeneous.**

Provided $a_1(x) \neq 0$ on the interval I of definition of a solution, we can write the **standard form** of the equation

$$\frac{dy}{dx} + P(x)y = f(x).$$

We'll be interested in equations (and intervals I) for which P and f are continuous on I.

Solutions (the General Solution)

$$\frac{dy}{dx} + P(x)y = f(x).$$

It turns out the solution will always have a basic form of $y = y_c + y_p$ where

- y_c is called the **complementary** solution and would solve the problem
$$\frac{dy}{dx} + P(x)y = 0$$
(called the associated homogeneous equation), and
- y_p is called the **particular** solution, and is heavily influenced by the function $f(x)$.

Motivating Example

$$x^2\frac{dy}{dx}+2xy=e^x$$

This is not in standard form, but we'll accept that for now.

Note that the left side happens to be $\frac{d}{dx}\left[x^2y\right]$. So our equation is

$$\frac{d}{dx}\left[x^2y\right]=e^x$$

Integrate both sides

$$\int\frac{d}{dx}\left[x^2y\right]dx=\int e^x\,dx$$

antiderivative of the derivative

$$x^2 y = e^x + C$$

Isolate y $\qquad y = \dfrac{e^x + C}{x^2}$ $\qquad$ The solutions

$$y = \underbrace{\frac{e^x}{x^2}}_{y_p} + \underbrace{\frac{C}{x^2}}_{y_c}$$

Derivation of Solution via Integrating Factor

Solve the equation in standard form

$$\frac{dy}{dx} + P(x)y = f(x)$$

We'll look for a positive function $\mu(x)$ such that when we multiply

$$\mu \frac{dy}{dx} + \mu P(x) y = \mu f(x)$$

the left side becomes one term $\frac{d}{dx}[\mu y]$.

We need

$$\frac{d}{dx}[\mu y] = \mu \frac{dy}{dx} + \frac{d\mu}{dx} y = \mu \frac{dy}{dx} + \mu P(x) y$$

must also match

match

We need μ to solve

$$\frac{d\mu}{dx} y = \mu P(x) y \implies \frac{d\mu}{dx} = \mu P(x)$$

this is separable!

$$\frac{d\mu}{dx} = \mu P(x) \implies \frac{1}{\mu}\frac{d\mu}{dx} = P(x)$$

$$\int \frac{1}{\mu}\frac{d\mu}{dx}\,dx = \int P(x)\,dx$$

$$\int \frac{1}{\mu}\,d\mu = \int P(x)\,dx$$

$$\ln \mu = \int P(x)\,dx$$

Exponentiate $\mu = e^{\int P(x)\,dx}$ called an integrating factor

Then $\mu \frac{dy}{dx} + \mu P y = \mu f$

$$\Rightarrow \frac{d}{dx}\left[\mu y\right] = \mu f$$

Integrate $\int \frac{d}{dx}\left[\mu y\right] dx = \int \mu f\, dx$

$$\mu y = \int \mu f(x)\, dx$$

Isolate y:

$$y = \frac{1}{\mu}\int \mu(x) f(x)\,dx$$

$$= \frac{1}{\mu}\left(\int \mu(x) f(x)\,dx + C\right)$$

since $\mu = e^{\int P(x)\,dx}$

$$y = \underbrace{e^{-\int P(x)\,dx}\int e^{\int P(x)\,dx} f(x)\,dx}_{y_p} + \underbrace{C\,e^{-\int P(x)\,dx}}_{y_c}$$

General Solution of First Order Linear ODE

- Put the equation in standard form $y' + P(x)y = f(x)$, and correctly identify the function $P(x)$.
- Obtain the integrating factor $\mu(x) = \exp\left(\int P(x)\,dx\right)$.
- Multiply both sides of the equation (in standard form) by the integrating factor μ. The left hand side **will always** collapse into the derivative of a product

$$\frac{d}{dx}[\mu(x)y] = \mu(x)f(x).$$

- Integrate both sides, and solve for y.

$$y(x) = \frac{1}{\mu(x)}\int \mu(x)f(x)\,dx = e^{-\int P(x)\,dx}\left(\int e^{\int P(x)\,dx}f(x)\,dx + C\right)$$

Solve the ODE

$$\frac{dy}{dx} + y = 3xe^{-x}$$

This is in standard form with

$P(x) = 1$

$$\mu = e^{\int P(x)\,dx} = e^{\int dx} = e^{x+c} = e^{x} \cdot e^{c} = Ae^{x}$$

Mult. by μ

$$Ae^{x}\frac{dy}{dx} + Ae^{x}y = Ae^{x}\left(3xe^{-x}\right)$$

$$A\left(e^{x}\frac{dy}{dx} + e^{x}y\right) = A\left(3xe^{-x}e^{x}\right)$$

A can cancel which is the same as letting $c = 0$

$$e^{x}\frac{dy}{dx} + e^{x} y = 3x$$

$$\frac{d}{dx}\left[e^{x} y\right] = 3x$$

$$\int \frac{d}{dx}\left[e^{x} y\right] dx = \int 3x \, dx$$

$$e^{x} y = 3\frac{x^2}{2} + C$$

$$e^{x} y = \frac{3}{2}x^2 + C \implies y = \frac{\frac{3}{2}x^2 + C}{e^{x}}$$

$$y = \frac{3}{2}x^2 e^{-x} + C e^{-x}$$

Solve the IVP

$$x\frac{dy}{dx} - y = 2x^2, \; x > 0 \quad y(1) = 5$$

The ODE is not in standard form.

In standard form, the eqn is

$$\frac{dy}{dx} - \frac{1}{x}y = \frac{2x^2}{x} = 2x \implies P(x) = \frac{-1}{x}$$

$$\mu = e^{\int P(x)\,dx} = e^{\int \frac{-1}{x}\,dx} = e^{-\ln x} = e^{\ln x^{-1}} = x^{-1} = \frac{1}{x}$$

$$\frac{1}{x}\frac{dy}{dx} - \frac{1}{x^2}y = 2x \cdot \frac{1}{x} = 2$$

$$\frac{d}{dx}\left[\frac{1}{x}y\right] = 2$$

$$\int \frac{d}{dx}\left[\frac{1}{x}y\right]dx = \int 2\,dx$$

$$\frac{1}{x}y = 2x + C$$

Mult by x

$$y = 2x^2 + Cx$$

Impose $y(1) = 5$ $\quad y(1) = 2(1^2) + C(1) = 5$

$$2 + C = 5 \implies C = 3$$

The solution to the IVP is

$$y = 2x^2 + 3x.$$

Verify

Just for giggles, lets verify that our solution $y = 2x^2 + 3x$ really does solve the differential equation we started with

$$x\frac{dy}{dx} - y = 2x^2.$$

$y = 2x^2 + 3x$

$y' = 4x + 3$

$$x\frac{dy}{dx} - y = x(4x+3) - (2x^2+3x)$$

$$= 4x^2 + \cancel{3x} - 2x^2 - \cancel{3x}$$

$$= 4x^2 - 2x^2$$

$$= 2x^2$$

as expected!

Steady and Transient States

For some linear equations, the term y_c decays as x (or t) grows. For example

Typo

$$\frac{dy}{dx} + y = 3xe^{-x} \quad \text{has solution} \quad y = \frac{3}{2}x^2 e^{-x} + Ce^{-x}.$$

$$\text{Here,} \quad y_p = \frac{3}{2}x^2 e^{-x} \text{ and } \quad y_c = Ce^{-x}.$$

Such a decaying complementary solution is called a **transient state.**

The corresponding particular solution is called a **steady state.**