August 24 Math 3260 sec. 57 Fall 2017

Section 1.4: The Matrix Equation $A\mathbf{x} = \mathbf{b}$.

For $m \times n$ matrix $A = [\mathbf{a}_1\ \mathbf{a}_2\ \cdots\ \mathbf{a}_n]$ and vector $\mathbf{x}$ in $\mathbb{R}^n$, we defined the product

$$A\mathbf{x} = x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n$$

which is a vector in $\mathbb{R}^m$.

Theorem

If A is the $m \times n$ matrix whose columns are the vectors $\mathbf{a}_1, \mathbf{a}_2, \cdots, \mathbf{a}_n$, and $\mathbf{b}$ is in $\mathbb{R}^m$, then the matrix equation

$$A\mathbf{x} = \mathbf{b}$$

has the same solution set as the vector equation

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n = \mathbf{b}$$

which, in turn, has the same solution set as the linear system of equations whose augmented matrix is

$$[\mathbf{a}_1 \quad \mathbf{a}_2 \quad \cdots \quad \mathbf{a}_n \quad \mathbf{b}].$$

That is, the equation $A\mathbf{x} = \mathbf{b}$ has a solution if and only if $\mathbf{b}$ is in $\mathrm{Span}\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n\}$.

Theorem

Let A be an $m \times n$ matrix. Then the following are logically equivalent (i.e. they are either all true or are all false).

(a) For each $\mathbf{b}$ in $\mathbb{R}^m$, the equation $A\mathbf{x} = \mathbf{b}$ has a solution.

(b) Each $\mathbf{b}$ in $\mathbb{R}^m$ is a linear combination of the columns of A.

(c) The columns of A span $\mathbb{R}^m$.

(d) A has a pivot position in every row.

A Scalar Product

If **u** and **v** are vectors in $\mathbb{R}^n$, we define a scalar product (also called the *dot* product) via

$$\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2 + \cdots + u_n v_n$$

Computing $A\mathbf{x}$

We can use a *row-vector* dot product rule. The i^{th} entry is $A\mathbf{x}$ is the sum of products of corresponding entries from row i of A with those of $\mathbf{x}$. For example

$$\begin{bmatrix} 1 & 0 & -3 \\ -2 & -1 & 4 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1(2) + 0(1) + (-3)(-1) \\ -2(2) + (-1)(1) + 4(-1) \end{bmatrix}$$

$$= \begin{bmatrix} 5 \\ -9 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 4 \\ -1 & 1 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} -3 \\ 2 \end{bmatrix} = \begin{bmatrix} 2(-3)+4(2) \\ -1(-3)+1(2) \\ 0(-3)+3(2) \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ 6 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1(x_1)+0(x_2)+0(x_3) \\ 0(x_1)+1(x_2)+0(x_3) \\ 0(x_1)+0(x_2)+1(x_3) \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Identity Matrix

We'll call an $n \times n$ matrix with 1's on the diagonal and 0's everywhere else—i.e. one that looks like

$$\begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

the $n \times n$ **identity** matrix and denote it by I_n. (We'll drop the subscript if it's obvious from the context.)

This matrix has the property that for each $\mathbf{x}$ in $\mathbb{R}^n$

$$I_n\mathbf{x} = \mathbf{x}.$$

Theorem: Properties of the Matrix Product

If A is an $m \times n$ matrix, $\mathbf{u}$ and $\mathbf{v}$ are vectors in $\mathbb{R}^n$, and c is any scalar, then

(a) $A(\mathbf{u}+\mathbf{v}) = A\mathbf{u} + A\mathbf{v}$, and

(b) $A(c\mathbf{u}) = cA\mathbf{u}$.

Section 1.5: Solution Sets of Linear Systems

Definition A linear system is said to be **homogeneous** if it can be written in the form

$$A\mathbf{x} = \mathbf{0}$$

for some $m \times n$ matrix A and where $\mathbf{0}$ is the zero vector in $\mathbb{R}^m$.

Theorem: A homogeneous system $A\mathbf{x} = \mathbf{0}$ always has at least one solution $\mathbf{x} = \mathbf{0}$.

The solution $\mathbf{x} = \mathbf{0}$ is called the **trivial solution**. A more interesting question for a homogeneous system is

Does it have a nontrivial solution?

Theorem

The homogeneous equation $A\mathbf{x} = \mathbf{0}$ has a nontrivial solution if and only if the system has at least one free variable.

Example: Determine if the homogeneous system has a nontrivial solution. Describe the solution set.

(a)
$$\begin{array}{rcrcl} 2x_1 & + & x_2 & = & 0 \\ x_1 & - & 3x_2 & = & 0 \end{array}$$

Augmented matrix

$$\begin{bmatrix} 2 & 1 & 0 \\ 1 & -3 & 0 \end{bmatrix}$$

$R_1 \leftrightarrow R_2$
$$\begin{bmatrix} 1 & -3 & 0 \\ 2 & 1 & 0 \end{bmatrix}$$
$-2R_1 + R_2 \to R_2$

$$\begin{bmatrix} 1 & -3 & 0 \\ 0 & 7 & 0 \end{bmatrix} \qquad \begin{array}{l} \frac{1}{7}R_2 \to R_2 \\ \text{then} \quad 3R_2 + R_1 \to R_1 \end{array}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \qquad \left.\begin{array}{l} \text{2 variables} \\ \text{2 pivot columns} \end{array}\right\} \Rightarrow \begin{array}{l} \text{no} \\ \text{free} \\ \text{variables} \end{array}$$

This system has only the trivial solution.

$$\text{(b)}\quad \begin{array}{rcrcrcr} 3x_1 & + & 5x_2 & - & 4x_3 & = & 0 \\ -3x_1 & - & 2x_2 & + & 4x_3 & = & 0 \\ 6x_1 & + & x_2 & - & 8x_3 & = & 0 \end{array}$$

Note that

$$\text{rref}\left(\begin{bmatrix} 3 & 5 & -4 \\ -3 & -2 & 4 \\ 6 & 1 & -8 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 & -\frac{4}{3} \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

and

$$\text{rref}\left(\begin{bmatrix} 3 & 5 & -4 & 0 \\ -3 & -2 & 4 & 0 \\ 6 & 1 & -8 & 0 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 & -\frac{4}{3} & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

For a homogeneous system (and only a homogeneous system) row reduction performed on the coefficient matrix is sufficient to determine solutions.

$$\left.\begin{array}{rcl} x_1 - \frac{4}{3}x_3 &=& 0 \\ x_2 &=& 0 \\ 0 &=& 0 \end{array}\right\} \Rightarrow \begin{array}{l} x_1 = \frac{4}{3}x_3 \\ x_2 = 0 \\ x_3 - \text{free} \end{array}$$

Solutions

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{4}{3}x_3 \\ 0 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} \frac{4}{3} \\ 0 \\ 1 \end{bmatrix} \quad \text{any } x_3 \text{ in } \mathbb{R}$$

All solutions $\vec{x}$ are in $\text{Span}\left\{ \begin{bmatrix} 4/3 \\ 0 \\ 1 \end{bmatrix} \right\}$.

(c) $x_1 - 2x_2 + 5x_3 = 0$

$$\begin{bmatrix} 1 & -2 & 5 & 0 \end{bmatrix}$$

$$\Rightarrow \quad x_1 = 2x_2 - 5x_3$$

x_2 - free

x_3 - free

All solutions $\vec{x}$ have the form

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2x_2 - 5x_3 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2x_2 \\ x_2 \\ 0 \end{bmatrix} + \begin{bmatrix} -5x_3 \\ 0 \\ x_3 \end{bmatrix}$$

$$= x_2 \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -5 \\ 0 \\ 1 \end{bmatrix} \qquad x_2, x_3 \text{ any } \mathbb{R}$$

This also says all solutions are in

$$\text{Span}\left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -5 \\ 1 \\ 0 \end{bmatrix} \right\}.$$

Parametric Vector Form of a Solution Set

Example (b) had a solution set consisting of vectors of the form $\mathbf{x} = x_3\mathbf{u}$. Example (c)'s solution set consisted of vector that look like $\mathbf{x} = x_2\mathbf{u} + x_3\mathbf{v}$. Since these are **linear combinations**, we could write the solution sets like

$$\text{Span}\{\mathbf{u}\} \quad \text{or} \quad \text{Span}\{\mathbf{u}, \mathbf{v}\}.$$

Instead of using the variables x_2 and/or x_3 we often substitute **parameters** such as s or t.
The forms

$$\mathbf{x} = s\mathbf{u}, \quad \text{or} \quad \mathbf{x} = s\mathbf{u} + t\mathbf{v}$$

are called **parametric vector forms.**

Example

The **parametric vector form** of the solution set of
$x_1 - 2x_2 + 5x_3 = 0$ is

$$\mathbf{x} = s\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -5 \\ 0 \\ 1 \end{bmatrix}, \quad \text{where } s, t \in \mathbb{R}.$$

Question: What geometric object is that solution set?

It's the plane in $\mathbb{R}^3$ containing $\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$, $\begin{bmatrix} -5 \\ 0 \\ 1 \end{bmatrix}$

and the origin.

Nonhomogeneous Systems

Find all solutions of the nonhomogeneous system of equations

$$\begin{array}{rcrcrcr} 3x_1 & + & 5x_2 & - & 4x_3 & = & 7 \\ -3x_1 & - & 2x_2 & + & 4x_3 & = & -1 \\ 6x_1 & + & x_2 & - & 8x_3 & = & -4 \end{array}$$

Using technology

$$\text{rref}\left(\begin{bmatrix} 3 & 5 & -4 & 7 \\ -3 & -2 & 4 & -1 \\ 6 & 1 & -8 & -4 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 & -\frac{4}{3} & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

$$\left.\begin{array}{l} x_1 \quad -\frac{4}{3}x_3 = -1 \\ \quad x_2 \quad = 2 \end{array}\right\} \Rightarrow \begin{array}{l} x_1 = -1 + \frac{4}{3}x_3 \\ x_2 = 2 \\ x_3 - \text{free} \end{array}$$

Solutions

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 + \frac{4}{3}x_3 \\ 2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} + \begin{bmatrix} \frac{4}{3}x_3 \\ 0 \\ x_3 \end{bmatrix}$$

$$= \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 4/3 \\ 0 \\ 1 \end{bmatrix}$$

Solutions of Nonhomogeneous Systems

Note that the solution in this example has the form

$$\mathbf{x} = \mathbf{p} + t\mathbf{v}$$

with $\mathbf{p}$ and $\mathbf{v}$ fixed vectors and t a varying parameter. Also note that the $t\mathbf{v}$ part is the solution to the previous example with the right hand side all zeros. This is no coincidence!

$\mathbf{p}$ is called a **particular solution**, and $t\mathbf{v}$ is called a solution to the associated homogeneous equation.

Theorem

Suppose the equation $A\mathbf{x} = \mathbf{b}$ is consistent for a given $\mathbf{b}$. Let $\mathbf{p}$ be a solution. Then the solution set of $A\mathbf{x} = \mathbf{b}$ is the set of all vectors of the form

$$\mathbf{x} = \mathbf{p} + \mathbf{v}_h,$$

where $\mathbf{v}_h$ is any solution of the associated homogeneous equation $A\mathbf{x} = \mathbf{0}$.

We can use a row reduction technique to get all parts of the solution in one process.

Example

Find the solution set of the following system. Express the solution set in parametric vector form.

$$\begin{array}{rcrcrcrcr} x_1 & + & x_2 & - & 2x_3 & + & 4x_4 & = & 1 \\ 2x_1 & + & 3x_2 & - & 6x_3 & + & 12x_4 & = & 4 \end{array}$$

$$\begin{bmatrix} 1 & 1 & -2 & 4 & 1 \\ 2 & 3 & -6 & 12 & 4 \end{bmatrix}$$

$-2R_1 + R_2 \to R_2$

$$\begin{array}{ccccc} -2 & -2 & 4 & -8 & -2 \\ 2 & 3 & -6 & 12 & 4 \end{array}$$

$$\begin{bmatrix} 1 & 1 & -2 & 4 & 1 \\ 0 & 1 & -2 & 4 & 2 \end{bmatrix}$$

$-R_2 + R_1 \to R_1$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & -1 \\ 0 & 1 & -2 & 4 & 2 \end{bmatrix}$$

$x_1 = -1$

$x_2 = 2x_3 - 4x_4 + 2$

$= 2 + 2x_3 - 4x_4$

x_3, x_4 - free

$$\vec{x} = \begin{bmatrix} -1 \\ 2 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 2 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ -4 \\ 0 \\ 1 \end{bmatrix}$$

$$\vec{x} = \begin{bmatrix} -1 \\ 2 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} 0 \\ 2 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ -4 \\ 0 \\ 1 \end{bmatrix} \qquad s, t \in \mathbb{R}$$