Section 1.4: The Matrix Equation $Ax = b$.

For $m \times n$ matrix $A = [a_1 \ a_2 \ \cdots \ a_n]$ and vector $x$ in $\mathbb{R}^n$, we defined the product

$$Ax = x_1a_1 + x_2a_2 + \cdots + x_na_n$$

which is a vector in $\mathbb{R}^m$. 
**Theorem**

If $A$ is the $m \times n$ matrix whose columns are the vectors $a_1, a_2, \ldots, a_n$, and $b$ is in $\mathbb{R}^m$, then the matrix equation

$$Ax = b$$

has the same solution set as the vector equation

$$x_1a_1 + x_2a_2 + \cdots + x_na_n = b$$

which, in turn, has the same solution set as the linear system of equations whose augmented matrix is

$$[a_1 \ a_2 \ \cdots \ a_n \ b].$$

That is, the equation $Ax = b$ has a solution if and only if $b$ is in $\text{Span}\{a_1, a_2, \ldots, a_n\}$.
Theorem

Let $A$ be an $m \times n$ matrix. Then the following are logically equivalent (i.e. they are either all true or are all false).

(a) For each $b$ in $\mathbb{R}^m$, the equation $Ax = b$ has a solution.

(b) Each $b$ in $\mathbb{R}^m$ is a linear combination of the columns of $A$.

(c) The columns of $A$ span $\mathbb{R}^m$.

(d) $A$ has a pivot position in every row.
A Scalar Product

If $\mathbf{u}$ and $\mathbf{v}$ are vectors in $\mathbb{R}^n$, we define a scalar product (also called the *dot* product) via

$$\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2 + \cdots + u_n v_n$$
Computing $A\mathbf{x}$

We can use a row-vector dot product rule. The $i^{th}$ entry is $A\mathbf{x}$ is the sum of products of corresponding entries from row $i$ of $A$ with those of $\mathbf{x}$. For example

$$
\begin{bmatrix}
1 & 0 & -3 \\
-2 & -1 & 4
\end{bmatrix}
\begin{bmatrix}
2 \\
1 \\
-1
\end{bmatrix}
= 
\begin{bmatrix}
1 \cdot (2) + 0 \cdot (1) + (-3) \cdot (-1) \\
-2 \cdot (2) + (-1) \cdot (1) + 4 \cdot (-1)
\end{bmatrix}
= 
\begin{bmatrix}
5 \\
-9
\end{bmatrix}
$$
\[
\begin{bmatrix}
2 & 4 \\
-1 & 1 \\
0 & 3
\end{bmatrix}
\begin{bmatrix}
-3 \\
2
\end{bmatrix}
= \begin{bmatrix}
(2)(-3) + 4(2) \\
(-1)(-3) + 1(2) \\
0(-3) + 3(2)
\end{bmatrix}
= \begin{bmatrix}
2 \\
5 \\
6
\end{bmatrix}
\]

\[
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3
\end{bmatrix}
= \begin{bmatrix}
x_1 + 0(x_2) + 0(x_3) \\
0(x_1) + 1(x_2) + 0(x_3) \\
0(x_1) + 0(x_2) + 1(x_3)
\end{bmatrix}
= \begin{bmatrix}
x_1 \\
x_2 \\
x_3
\end{bmatrix}
\]
Identity Matrix

We’ll call an \( n \times n \) matrix with 1’s on the diagonal and 0’s everywhere else—i.e. one that looks like

\[
\begin{bmatrix}
1 & 0 & \cdots & 0 \\
0 & 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 1
\end{bmatrix}
\]

the \( n \times n \) identity matrix and denote it by \( I_n \). (We’ll drop the subscript if it’s obvious from the context.)

This matrix has the property that for each \( \mathbf{x} \) in \( \mathbb{R}^n \)

\[
I_n \mathbf{x} = \mathbf{x}.
\]
Theorem: Properties of the Matrix Product

If $A$ is an $m \times n$ matrix, $\mathbf{u}$ and $\mathbf{v}$ are vectors in $\mathbb{R}^n$, and $c$ is any scalar, then

(a) $A(\mathbf{u} + \mathbf{v}) = A\mathbf{u} + A\mathbf{v}$, and

(b) $A(c\mathbf{u}) = cA\mathbf{u}$. 
Section 1.5: Solution Sets of Linear Systems

**Definition** A linear system is said to be **homogeneous** if it can be written in the form

\[ Ax = 0 \]

for some \( m \times n \) matrix \( A \) and where \( 0 \) is the zero vector in \( \mathbb{R}^m \).

**Theorem:** A homogeneous system \( Ax = 0 \) always has at least one solution \( x = 0 \).

The solution \( x = 0 \) is called the **trivial solution**. A more interesting question for a homogeneous system is

**Does it have a nontrivial solution?**
Theorem
The homogeneous equation $Ax = 0$ has a nontrivial solution if and only if the system has at least one free variable.

Example: Determine if the homogeneous system has a nontrivial solution. Describe the solution set.

(a) $2x_1 + x_2 = 0$
$x_1 - 3x_2 = 0$

The augmented matrix is
\[
\begin{bmatrix}
2 & 1 & 0 \\
1 & -3 & 0
\end{bmatrix}
\]

\[
\begin{bmatrix}
1 & -3 & 0 \\
2 & 1 & 0
\end{bmatrix}
\]

$r_1 \leftrightarrow r_2$

$-2r_1 + r_2 \rightarrow r_2$

\[
\begin{bmatrix}
1 & -3 & 0 \\
0 & 7 & 0
\end{bmatrix}
\]
\[ \frac{1}{7} R_2 \rightarrow R_2 \]

then

\[ 3R_2 + R_1 \rightarrow R_1 \]

\[ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \]

2 variables, 2 pivots \( \Rightarrow \) no free variables.

This system has only the trivial solution.
\[3x_1 + 5x_2 - 4x_3 = 0\]
\[-3x_1 - 2x_2 + 4x_3 = 0\]
\[6x_1 + x_2 - 8x_3 = 0\]

Note that

\[
\text{rref} \left( \begin{bmatrix} 3 & 5 & -4 \\ -3 & -2 & 4 \\ 6 & 1 & -8 \end{bmatrix} \right) = \begin{bmatrix} 1 & 0 & -\frac{4}{3} \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix},
\]

and

\[
\text{rref} \left( \begin{bmatrix} 3 & 5 & -4 & 0 \\ -3 & -2 & 4 & 0 \\ 6 & 1 & -8 & 0 \end{bmatrix} \right) = \begin{bmatrix} 1 & 0 & -\frac{4}{3} & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.
\]

For a homogeneous system (and only a homogeneous system) row reduction performed on the coefficient matrix is sufficient to determine solutions.
2 pivots, 3 variables $\Rightarrow$ one free variable

There are non-trivial solutions

\[
\begin{align*}
    x_1 & - \frac{4}{3} x_3 = 0 \\
    x_2 & = 0
\end{align*}
\]

$\Rightarrow$

\[
\begin{align*}
    x_1 & = \frac{4}{3} x_3 \\
    x_2 & = 0 \\
    x_3 & \text{ free}
\end{align*}
\]

we can write this as

\[
\begin{bmatrix}
    x_1 \\
    x_2 \\
    x_3
\end{bmatrix}
= \begin{bmatrix}
    \frac{4}{3} x_3 \\
    0 \\
    x_3
\end{bmatrix}
= x_3 \begin{bmatrix}
    \frac{4}{3} \\
    0 \\
    1
\end{bmatrix}
\]

where $x_3$ is in $\mathbb{R}$
we can also say the solution set is

\[
\text{Span } \left\{ \begin{bmatrix} 4 \\ 1 \\ 3 \end{bmatrix} \right\}.
\]
(c) \[ x_1 - 2x_2 + 5x_3 = 0 \]

\[
\begin{bmatrix} 1 & -2 & \leq & 0 \end{bmatrix}
\]

Solutions:
\[ x_1 = 2x_2 - 5x_3 \]
\[ x_2, x_3 \text{ - free} \]

All solutions \( \mathbf{x} \) have the form

\[
\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2x_2 - 5x_3 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2x_2 \\ x_2 \\ 0 \end{bmatrix} + \begin{bmatrix} -5x_3 \\ 0 \\ x_3 \end{bmatrix}
\]

\[
= x_2 \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -5 \\ 0 \\ 1 \end{bmatrix}
\]
The solution set is

$$\text{Span} \left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -5 \\ 0 \\ 1 \end{bmatrix} \right\}.$$
Parametric Vector Form of a Solution Set

Example (b) had a solution set consisting of vectors of the form $x = x_3 \mathbf{u}$. Example (c)'s solution set consisted of vectors that look like $x = x_2 \mathbf{u} + x_3 \mathbf{v}$. Since these are linear combinations, we could write the solution sets like

$$\text{Span}\{\mathbf{u}\} \quad \text{or} \quad \text{Span}\{\mathbf{u}, \mathbf{v}\}.$$  

Instead of using the variables $x_2$ and/or $x_3$ we often substitute parameters such as $s$ or $t$. The forms

$$x = s \mathbf{u}, \quad \text{or} \quad x = s \mathbf{u} + t \mathbf{v}$$

are called parametric vector forms.
Example

The parametric vector form of the solution set of
\[ x_1 - 2x_2 + 5x_3 = 0 \]
is
\[
\mathbf{x} = s \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -5 \\ 0 \\ 1 \end{bmatrix}, \quad \text{where } s, t \in \mathbb{R}.
\]

Question: What geometric object is that solution set?

It's a plane in \( \mathbb{R}^3 \) containing \((2,1,0), (-5,0,1)\) and the origin.
Nonhomogeneous Systems

Find all solutions of the nonhomogeneous system of equations

\[
\begin{align*}
3x_1 &+ 5x_2 - 4x_3 = 7 \\
-3x_1 &- 2x_2 + 4x_3 = -1 \\
6x_1 &+ x_2 - 8x_3 = -4
\end{align*}
\]

Using technology

\[
\text{rref}\left(\begin{bmatrix} 3 & 5 & -4 & 7 \\ -3 & -2 & 4 & -1 \\ 6 & 1 & -8 & -4 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 & -\frac{4}{3} & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}.
\]

\[
\begin{align*}
x_1 - \frac{4}{3}x_3 &= -1 \\
x_2 &= 2 \\
x_3 &= \text{free}
\end{align*}
\]

\[
\begin{align*}
x_1 &= -1 + \frac{4}{3}x_3 \\
x_2 &= 2 \\
x_3 &= \text{free}
\end{align*}
\]
The solutions are

\[
\begin{bmatrix}
  x_1 \\
  x_2 \\
  x_3
\end{bmatrix} = \begin{bmatrix}
  -1 + \frac{4}{3} x_3 \\
  2 \\
  x_3
\end{bmatrix} = \begin{bmatrix}
  -1 \\
  2 \\
  0
\end{bmatrix} + \begin{bmatrix}
  \frac{4}{3} x_3 \\
  0 \\
  x_3
\end{bmatrix}
\]

\[
= \begin{bmatrix}
  -1 \\
  2 \\
  0
\end{bmatrix} + x_3 \begin{bmatrix}
  \frac{4}{3} \\
  0 \\
  1
\end{bmatrix}
\]

This is the line span \{"\begin{bmatrix} 0 \\
  1 \end{bmatrix}\}\) translated to pass through \(\begin{bmatrix}
  -1 \\
  2 \\
  0
\end{bmatrix}\).
Solutions of Nonhomogeneous Systems

Note that the solution in this example has the form

\[ x = p + tv \]

with \( p \) and \( v \) fixed vectors and \( t \) a varying parameter. Also note that the \( tv \) part is the solution to the previous example with the right hand side all zeros. This is no coincidence!

\( p \) is called a \textbf{particular solution}, and \( tv \) is called a solution to the associated homogeneous equation.
Theorem

Suppose the equation $Ax = b$ is consistent for a given $b$. Let $p$ be a solution. Then the solution set of $Ax = b$ is the set of all vectors of the form

$$x = p + v_h,$$

where $v_h$ is any solution of the associated homogeneous equation $Ax = 0$.

We can use a row reduction technique to get all parts of the solution in one process.
Example

Find the solution set of the following system. Express the solution set in parametric vector form.

\[ \begin{align*}
  x_1 &+ x_2 - 2x_3 + 4x_4 = 1 \\
  2x_1 &+ 3x_2 - 6x_3 + 12x_4 = 4
\end{align*} \]

\[
\begin{bmatrix}
  1 & 1 & -2 & 4 & 1 \\
  2 & 3 & -6 & 12 & 4
\end{bmatrix}
\]

\[
\begin{bmatrix}
  1 & 1 & -2 & 4 & 1 \\
  0 & 1 & -2 & 4 & 2
\end{bmatrix}
\]

-2R_1 + R_2 \rightarrow R_2

\[
\begin{bmatrix}
  1 & 1 & -2 & 4 & 1 \\
  0 & 1 & -2 & 4 & 2
\end{bmatrix}
\]

-2R_1 + R_2 \rightarrow R_2

\[
\begin{bmatrix}
  1 & 1 & -2 & 4 & 1 \\
  0 & 1 & -2 & 4 & 2
\end{bmatrix}
\]

- R_2 + R_1 \rightarrow R_1
\[
\begin{bmatrix}
1 & 0 & 0 & 0 & -1 \\
0 & 1 & -2 & 0 & 2
\end{bmatrix}
\]

\[
x_1 = -1
\]

\[
x_2 = 2 + 2x_3 - 4x_4
\]

\[
x_3, x_4 \text{ free}
\]

The solutions

\[
\hat{x} = \begin{bmatrix}
-1 \\
2 \\
0 \\
0
\end{bmatrix} + x_3 \begin{bmatrix}
0 \\
2 \\
1 \\
0
\end{bmatrix} + x_4 \begin{bmatrix}
0 \\
0 \\
0 \\
1
\end{bmatrix}
\]

\[
\hat{\chi} = \begin{bmatrix}
-1 \\
2 \\
0 \\
0
\end{bmatrix} + s \begin{bmatrix}
0 \\
2 \\
1 \\
0
\end{bmatrix} + \ell \begin{bmatrix}
0 \\
-4 \\
0 \\
1
\end{bmatrix}, \quad s, \ell \text{ in } \mathbb{R}
\]
Section 1.7: Linear Independence

We already know that a homogeneous equation $Ax = 0$ can be thought of as an equation in the column vectors of the matrix $A = [a_1 \ a_2 \ \cdots \ a_n]$ as

$$x_1a_1 + x_2a_2 + \cdots x_na_n = 0.$$ 

And, we know that at least one solution (the trivial one $x_1 = x_2 = \cdots = x_n = 0$) always exists.

Whether or not there is a nontrivial solution gives us a way to characterize the vectors $a_1, \ldots, a_n$. 
Definition: Linear Dependence/Independence

An indexed set of vectors \( \{v_1, v_2, \ldots, v_p\} \) in \( \mathbb{R}^n \) is said to be **linearly independent** if the vector equation

\[
x_1v_1 + x_2v_2 + \cdots + x_pv_p = 0
\]

has only the trivial solution.

The set \( \{v_1, v_2, \ldots, v_p\} \) is said to be **linearly dependent** if there exists a set of weights \( c_1, c_2, \ldots, c_p \) at least one of which is nonzero such that

\[
c_1v_1 + c_2v_2 + \cdots + c_pv_p = 0.
\]

(i.e. Provided the homogeneous equation possesses a nontrivial solution.)

An equation \( c_1v_1 + c_2v_2 + \cdots + c_pv_p = 0 \), with at least one \( c_i \neq 0 \), is called a **linear dependence relation**.
Special Cases

A set with two vectors \( \{ \mathbf{v}_1, \mathbf{v}_2 \} \) is linearly dependent if one is a scalar multiple of the other.

If they are linearly dependent, then there exists \( c_1, c_2 \) not both zero such that

\[
c_1 \mathbf{v}_1 + c_2 \mathbf{v}_2 = \mathbf{0}
\]

We can assume \( c_1 \neq 0 \) (else relabel \( \mathbf{v}_1 \) and \( \mathbf{v}_2 \))

\[
c_1 \mathbf{v}_1 = -c_2 \mathbf{v}_2 \quad \Rightarrow \quad \mathbf{v}_1 = -\frac{c_2}{c_1} \mathbf{v}_2 = k \mathbf{v}_2
\]

where \( k = -\frac{c_2}{c_1} \)
Example

Determine if the set is linearly dependent or linearly independent.

(a)  \( \mathbf{v}_1 = \begin{bmatrix} 2 \\ 4 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} -1 \\ -2 \end{bmatrix} \)

Dependent: \( \mathbf{v}_1 = -2 \mathbf{v}_2 \)

(b)  \( \mathbf{v}_1 = \begin{bmatrix} 2 \\ 4 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} -1 \\ 2 \end{bmatrix} \)

Lin. independent.