#### August 24 Math 3260 sec. 58 Fall 2017

#### Section 1.4: The Matrix Equation Ax = b.

For  $m \times n$  matrix  $A = [\mathbf{a}_1 \ \mathbf{a}_2 \ \cdots \ \mathbf{a}_n]$  and vector  $\mathbf{x}$  in  $\mathbb{R}^n$ , we defined the product

$$A\mathbf{x} = x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n$$

which is a vector in  $\mathbb{R}^m$ .

#### Theorem

If *A* is the  $m \times n$  matrix whose columns are the vectors  $\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n$ , and **b** is in  $\mathbb{R}^m$ , then the matrix equation

$$A\mathbf{x} = \mathbf{b}$$

has the same solution set as the vector equation

$$x_1a_1 + x_2a_2 + \cdots + x_na_n = b$$

which, in turn, has the same solution set as the linear system of equations whose augmented matrix is

$$[\mathbf{a}_1 \ \mathbf{a}_2 \ \cdots \ \mathbf{a}_n \ \mathbf{b}].$$

That is, the equation  $A\mathbf{x} = \mathbf{b}$  has a solution if and only if **b** is in Span{ $\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n$  }.

#### Theorem

Let *A* be an  $m \times n$  matrix. Then the following are logically equivalent (i.e. they are either all true or are all false).

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August 23, 2017

3/39

- (a) For each **b** in  $\mathbb{R}^m$ , the equation  $A\mathbf{x} = \mathbf{b}$  has a solution.
- (b) Each **b** in  $\mathbb{R}^m$  is a linear combination of the columns of *A*.
- (c) The columns of A span  $\mathbb{R}^m$ .
- (d) A has a pivot position in every row.

## A Scalar Product

If **u** and **v** are vectors in  $\mathbb{R}^n$ , we define a scalar product (also called the *dot* product) via

$$\mathbf{u}\cdot\mathbf{v}=u_1v_1+u_2v_2+\cdots+u_nv_n$$

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# Computing Ax

We can use a *row-vector* dot product rule. The  $i^{th}$  entry is  $A\mathbf{x}$  is the sum of products of corresponding entries from row i of A with those of  $\mathbf{x}$ . For example

2

August 23, 2017

5/39

$$\begin{bmatrix} 1 & 0 & -3 \\ -2 & -1 & 4 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 & (2) + O(1) + (-3) & (-1) \\ -2(2) + (-1) & (1) + 4 & (-1) \end{bmatrix}$$

$$\left[\begin{array}{rrr} 2 & 4 \\ -1 & 1 \\ 0 & 3 \end{array}\right] \left[\begin{array}{r} -3 \\ 2 \end{array}\right] =$$

$$\begin{bmatrix} 2 (-3) + 4 (2) \\ -1 (-3) + 1 (2) \\ 0 (-3) + 3 (2) \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ 6 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1(x_1) + O(x_2) + O(x_3) \\ O(x_1) + I(x_2) + O(x_3) \\ O(x_1) + O(x_2) + I(x_3) \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

# **Identity Matrix**

We'll call an  $n \times n$  matrix with 1's on the diagonal and 0's everywhere else—i.e. one that looks like

$$\begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

the  $n \times n$  **identity** matrix and denote it by  $I_n$ . (We'll drop the subscript if it's obvious from the context.)

This matrix has the property that for each  $\mathbf{x}$  in  $\mathbb{R}^n$ 

$$l_n \mathbf{x} = \mathbf{x}.$$

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August 23, 2017

7/39

## Theorem: Properties of the Matrix Product

If *A* is an  $m \times n$  matrix, **u** and **v** are vectors in  $\mathbb{R}^n$ , and *c* is any scalar, then

(a)  $A(\mathbf{u} + \mathbf{v}) = A\mathbf{u} + A\mathbf{v}$ , and

(b)  $A(c\mathbf{u}) = cA\mathbf{u}$ .

# Section 1.5: Solution Sets of Linear Systems

**Definition** A linear system is said to be **homogeneous** if it can be written in the form

$$A\mathbf{x} = \mathbf{0}$$

for some  $m \times n$  matrix A and where **0** is the zero vector in  $\mathbb{R}^m$ .

**Theorem:** A homogeneous system  $A\mathbf{x} = \mathbf{0}$  always has at least one solution  $\mathbf{x} = \mathbf{0}$ .

The solution  $\mathbf{x} = \mathbf{0}$  is called the **trivial solution**. A more interesting question for a homogeneous system is

#### Does it have a nontrivial solution?

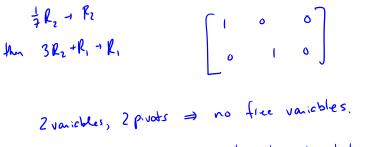
## Theorem

The homogeneous equation  $A\mathbf{x} = \mathbf{0}$  has a nontrivial solution if and only if the system has at least one free variable.

**Example:** Determine if the homogeneous system has a nontrivial solution. Describe the solution set.

(a) 
$$\begin{array}{l} 2x_1 + x_2 = 0 \\ x_1 - 3x_2 = 0 \end{array}$$
The organized metric  $\begin{bmatrix} 2 & 1 & 0 \\ 1 & -3 & 0 \end{bmatrix}$ 

$$\begin{array}{l} \begin{bmatrix} 1 & -3 & 0 \\ 2 & 1 & 0 \end{bmatrix} \qquad \begin{array}{l} -2R_1 + R_2 \rightarrow R_2 \\ \begin{bmatrix} 1 & -3 & 0 \\ 0 & 7 & 0 \end{bmatrix}$$



This system has only the trivial solution.

rref 
$$\begin{pmatrix} 3 \\ -3 \\ -3 \end{pmatrix}$$

and

Noto that

$$\operatorname{rref}\left(\left[\begin{array}{rrr} -3 & -2 & 4 \\ 6 & 1 & -8 \end{array}\right]\right) = \left[\begin{array}{rrr} 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right],$$
$$\operatorname{rref}\left(\left[\begin{array}{rrr} 3 & 5 & -4 & 0 \\ -3 & -2 & 4 & 0 \\ 6 & 1 & -8 & 0 \end{array}\right]\right) = \left[\begin{array}{rrr} 1 & 0 & -\frac{4}{3} & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

 $5 -4 \rceil \setminus \begin{bmatrix} 1 & 0 & -\frac{4}{2} \end{bmatrix}$ 

For a homogeneous system (and only a homogeneous system) row reduction performed on the coefficient matrix is sufficient to determine solutions.

2 pivets, 3 variables ⇒ one free variable There are non trivial solutions  $\begin{array}{ccc} \chi_{1} & -\frac{4}{3}\chi_{3} = 0 \\ \chi_{2} & z \end{array} \begin{array}{c} \chi_{1} = \frac{4}{3}\chi_{3} \\ \chi_{2} = 0 \end{array}$ Xa-fire be can write this as Solutions  $\begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = \begin{bmatrix} \frac{4}{3} \times 3 \\ 0 \\ X_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$ where x3 is in TR

August 23, 2017 13 / 39

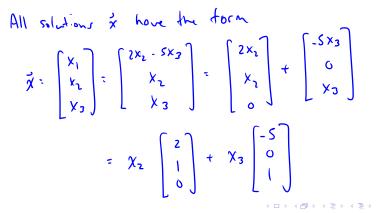
We can also say the solution set is  

$$Spon \left\{ \begin{bmatrix} 413\\0\\1 \end{bmatrix} \right\}.$$

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 August 23, 2017 14 / 39

(c) 
$$x_1 - 2x_2 + 5x_3 = 0$$
 [1 - 2 5 0]

Solutions 
$$\chi_1 = 2\chi_2 - 5\chi_3$$
  
 $\chi_2, \chi_3$  -free



The solution set is  
Span 
$$\left\{ \begin{bmatrix} 2\\1\\0 \end{bmatrix}, \begin{bmatrix} -5\\0\\1 \end{bmatrix} \right\}.$$

## Parametric Vector Form of a Solution Set

Example (b) had a solution set consisting of vectors of the form  $\mathbf{x} = x_3 \mathbf{U}$  Example (c)'s solution set consisted of vector that look like  $\mathbf{x} = x_2 \mathbf{u} + x_3 \mathbf{v}$ . Since these are linear combinations, we could write the solution sets like

Span{ $\mathbf{u}$ } or Span{ $\mathbf{u}, \mathbf{v}$ }.

Instead of using the variables  $x_2$  and/or  $x_3$  we often substitute parameters such as s or t.

The forms

$$\mathbf{x} = s\mathbf{u}$$
, or  $\mathbf{x} = s\mathbf{u} + t\mathbf{v}$ 

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17/39

are called parametric vector forms.

## Example

The **parametric vector form** of the solution set of  $x_1 - 2x_2 + 5x_3 = 0$  is

$$\mathbf{x} = s \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -5 \\ 0 \\ 1 \end{bmatrix}, \text{ where } s, t \in \mathbb{R}.$$

Question: What geometric object is that solution set?

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## Nonhomogeneous Systems

Find all solutions of the nonhomogeneous system of equations

#### Using technology

$$\operatorname{rref}\left(\left[\begin{array}{ccccc} 3 & 5 & -4 & 7 \\ -3 & -2 & 4 & -1 \\ 6 & 1 & -8 & -4 \end{array}\right]\right) = \left[\begin{array}{ccccc} 1 & 0 & -\frac{4}{3} & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right].$$
$$X_{1} \quad -\frac{4}{3}\chi_{3} = -1$$
$$\chi_{2} \quad z = 2$$
$$\begin{cases} \Rightarrow \quad \chi_{1} = -1 + \frac{4}{3}\chi_{3} \\ \chi_{2} = 2 \\ \chi_{3} = -1 \end{array}\right]$$

August 23, 2017 19 / 39

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The solutions are  

$$\begin{aligned}
x_{1} \\
x_{2} \\
x_{3} \\
x_{4} \\
x_{5} \\$$

August 23, 2017 20 / 39

# Solutions of Nonhomogeneous Systems

Note that the solution in this example has the form

 $\mathbf{x} = \mathbf{p} + t\mathbf{v}$ 

with **p** and **v** fixed vectors and *t* a varying parameter. Also note that the t**v** part is the solution to the previous example with the right hand side all zeros. This is no coincidence!

**p** is called a **particular solution**, and *t***v** is called a solution to the associated homogeneous equation.

August 23, 2017 22 / 39

#### Theorem

Suppose the equation  $A\mathbf{x} = \mathbf{b}$  is consistent for a given **b**. Let **p** be a solution. Then the solution set of  $A\mathbf{x} = \mathbf{b}$  is the set of all vectors of the form

$$\mathbf{x} = \mathbf{p} + \mathbf{v}_h,$$

where  $\mathbf{v}_h$  is any solution of the associated homogeneous equation  $A\mathbf{x} = \mathbf{0}$ .

We can use a row reduction technique to get all parts of the solution in one process.

August 23, 2017

23/39

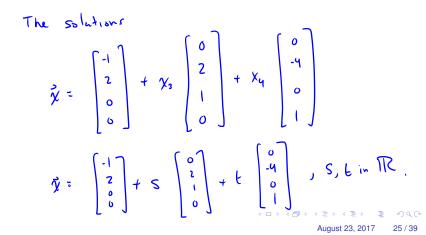
## Example

Find the solution set of the following system. Express the solution set in parametric vector form.

August 23, 2017 24 / 39

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## Section 1.7: Linear Independence

We already know that a homogeneous equation  $A\mathbf{x} = \mathbf{0}$  can be thought of as an equation in the column vectors of the matrix  $A = [\mathbf{a}_1 \ \mathbf{a}_2 \ \cdots \ \mathbf{a}_n]$  as

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n = \mathbf{0}.$$

August 23, 2017

27/39

And, we know that at least one solution (the trivial one  $x_1 = x_2 = \cdots = x_n = 0$  always exists.

Whether or not there is a nontrivial solution gives us a way to characterize the vectors  $\mathbf{a}_1, \ldots, \mathbf{a}_n$ .

## Definition: Linear Dependence/Independence

An indexed set of vectors  $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_p\}$  in  $\mathbb{R}^n$  is said to be **linearly independent** if the vector equation

$$x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \cdots + x_p\mathbf{v}_p = \mathbf{0}$$

has only the trivial solution.

The set  $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_p\}$  is said to be **linearly dependent** if there exists a set of weights  $c_1, c_2, \dots, c_p$  at least one of which is nonzero such that

$$c_1\mathbf{v}_1+c_2\mathbf{v}_2+\cdots c_p\mathbf{v}_p=\mathbf{0}.$$

(i.e. Provided the homogeneous equation possesses a nontrivial solution.)

An equation  $c_1 \mathbf{v}_1 + c_2 \mathbf{v}_2 + \cdots + c_p \mathbf{v}_p = \mathbf{0}$ , with at least one  $c_i \neq 0$ , is called a **linear dependence relation**.

#### **Special Cases**

A set with two vectors  $\{\bm{v}_1, \bm{v}_2\}$  is linearly dependent if one is a scalar multiple of the other.

If they are linearly dependent, then then exists  

$$C_1, C_2$$
 not both  $3e_0$  such that  
 $C_1, V_1 + (2V_2 = \vec{0})$   
We can assume  $C_1 \neq 0$  (also relabel  $V_1$  and  $V_2$ )  
 $C_1 V_1 = -C_2 V_2 \implies V_1 = -\frac{C_2}{C_1} V_2 = k V_2$   
Where  $k = -\frac{C_2}{C_1}$ 

August 23, 2017 29 / 39

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## Example

Determine if the set is linearly dependent or linearly independent.

(a) 
$$\mathbf{v}_1 = \begin{bmatrix} 2\\ 4 \end{bmatrix}$$
,  $\mathbf{v}_2 = \begin{bmatrix} -1\\ -2 \end{bmatrix}$  Dependent:  $\vec{v}_1 = -\vec{z} \vec{v}_2$ 

(b) 
$$\mathbf{v}_1 = \begin{bmatrix} 2\\4 \end{bmatrix}, \ \mathbf{v}_2 = \begin{bmatrix} -1\\2 \end{bmatrix}$$
 Lin. independent.

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