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Section 1.4: The Matrix Equation $A\mathbf{x} = \mathbf{b}$.

For $m \times n$ matrix $A = [\mathbf{a}_1\ \mathbf{a}_2\ \cdots\ \mathbf{a}_n]$ and vector $\mathbf{x}$ in $\mathbb{R}^n$, we defined the product

$$A\mathbf{x} = x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n$$

which is a vector in $\mathbb{R}^m$.

Theorem

If A is the $m \times n$ matrix whose columns are the vectors $\mathbf{a}_1, \mathbf{a}_2, \cdots, \mathbf{a}_n$, and $\mathbf{b}$ is in $\mathbb{R}^m$, then the matrix equation

$$A\mathbf{x} = \mathbf{b}$$

has the same solution set as the vector equation

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n = \mathbf{b}$$

which, in turn, has the same solution set as the linear system of equations whose augmented matrix is

$$[\mathbf{a}_1 \quad \mathbf{a}_2 \quad \cdots \quad \mathbf{a}_n \quad \mathbf{b}].$$

That is, the equation $A\mathbf{x} = \mathbf{b}$ has a solution if and only if $\mathbf{b}$ is in $\text{Span}\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n\}$.

Theorem

Let A be an $m \times n$ matrix. Then the following are logically equivalent (i.e. they are either all true or are all false).

(a) For each $\mathbf{b}$ in $\mathbb{R}^m$, the equation $A\mathbf{x} = \mathbf{b}$ has a solution.

(b) Each $\mathbf{b}$ in $\mathbb{R}^m$ is a linear combination of the columns of A.

(c) The columns of A span $\mathbb{R}^m$.

(d) A has a pivot position in every row.

A Scalar Product

If **u** and **v** are vectors in $\mathbb{R}^n$, we define a scalar product (also called the *dot* product) via

$$\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2 + \cdots + u_n v_n$$

Computing $A\mathbf{x}$

We can use a *row-vector* dot product rule. The i^{th} entry is $A\mathbf{x}$ is the sum of products of corresponding entries from row i of A with those of $\mathbf{x}$. For example

$$\begin{bmatrix} 1 & 0 & -3 \\ -2 & -1 & 4 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1(2) + 0(1) + (-3)(-1) \\ -2(2) + (-1)(1) + 4(-1) \end{bmatrix}$$

$$= \begin{bmatrix} 5 \\ -9 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 4 \\ -1 & 1 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} -3 \\ 2 \end{bmatrix} = \begin{bmatrix} 2(-3)+4(2) \\ -1(-3)+1(2) \\ 0(-3)+3(2) \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ 6 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1(x_1)+0(x_2)+0(x_3) \\ 0(x_1)+1(x_2)+0(x_3) \\ 0(x_1)+0(x_2)+1(x_3) \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Identity Matrix

We'll call an $n \times n$ matrix with 1's on the diagonal and 0's everywhere else—i.e. one that looks like

$$\begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

the $n \times n$ **identity** matrix and denote it by I_n. (We'll drop the subscript if it's obvious from the context.)

This matrix has the property that for each $\mathbf{x}$ in $\mathbb{R}^n$

$$I_n\mathbf{x} = \mathbf{x}.$$

Theorem: Properties of the Matrix Product

If A is an $m \times n$ matrix, $\mathbf{u}$ and $\mathbf{v}$ are vectors in $\mathbb{R}^n$, and c is any scalar, then

(a) $A(\mathbf{u} + \mathbf{v}) = A\mathbf{u} + A\mathbf{v}$, and

(b) $A(c\mathbf{u}) = cA\mathbf{u}$.

Section 1.5: Solution Sets of Linear Systems

Definition A linear system is said to be **homogeneous** if it can be written in the form

$$A\mathbf{x} = \mathbf{0}$$

for some $m \times n$ matrix A and where $\mathbf{0}$ is the zero vector in $\mathbb{R}^m$.

Theorem: A homogeneous system $A\mathbf{x} = \mathbf{0}$ always has at least one solution $\mathbf{x} = \mathbf{0}$.

The solution $\mathbf{x} = \mathbf{0}$ is called the **trivial solution**. A more interesting question for a homogeneous system is

Does it have a nontrivial solution?

Theorem

The homogeneous equation $A\mathbf{x} = \mathbf{0}$ has a nontrivial solution if and only if the system has at least one free variable.

Example: Determine if the homogeneous system has a nontrivial solution. Describe the solution set.

(a) $$\begin{array}{rcrcl} 2x_1 & + & x_2 & = & 0 \\ x_1 & - & 3x_2 & = & 0 \end{array}$$

The augmented matrix is

$$\begin{bmatrix} 2 & 1 & 0 \\ 1 & -3 & 0 \end{bmatrix}$$

$R_1 \leftrightarrow R_2$

$$\begin{bmatrix} 1 & -3 & 0 \\ 2 & 1 & 0 \end{bmatrix}$$

$-2R_1 + R_2 \to R_2$

$$\begin{bmatrix} 1 & -3 & 0 \\ 0 & 7 & 0 \end{bmatrix}$$

$\frac{1}{7}R_2 \to R_2$

then $3R_2 + R_1 \to R_1$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

2 variables, 2 pivots $\Rightarrow$ no free variables.

This system has only the trivial solution.

$$
\text{(b)} \quad \begin{array}{rcrcrcr}
3x_1 & + & 5x_2 & - & 4x_3 & = & 0 \\
-3x_1 & - & 2x_2 & + & 4x_3 & = & 0 \\
6x_1 & + & x_2 & - & 8x_3 & = & 0
\end{array}
$$

Note that

$$
\text{rref}\left(\begin{bmatrix} 3 & 5 & -4 \\ -3 & -2 & 4 \\ 6 & 1 & -8 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 & -\frac{4}{3} \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix},
$$

and

$$
\text{rref}\left(\begin{bmatrix} 3 & 5 & -4 & 0 \\ -3 & -2 & 4 & 0 \\ 6 & 1 & -8 & 0 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 & -\frac{4}{3} & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.
$$

For a homogeneous system (and only a homogeneous system) row reduction performed on the coefficient matrix is sufficient to determine solutions.

2 pivots, 3 variables $\Rightarrow$ one free variable

There are non trivial solutions

$$\left.\begin{array}{l} x_1 \qquad - \frac{4}{3}x_3 = 0 \\ \qquad x_2 \qquad\quad = 0 \end{array}\right\} \Rightarrow \begin{array}{l} x_1 = \frac{4}{3}x_3 \\ x_2 = 0 \\ x_3 - \text{free} \end{array}$$

We can write this as

Solutions

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{4}{3}x_3 \\ 0 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 4/3 \\ 0 \\ 1 \end{bmatrix}$$

where x_3 is in $\mathbb{R}$

we can also say the solution set is

$$\text{Span}\left\{\begin{bmatrix} 4/3 \\ 0 \\ 1 \end{bmatrix}\right\}.$$

(c) $x_1 - 2x_2 + 5x_3 = 0$

$$\begin{bmatrix} 1 & -2 & 5 & 0 \end{bmatrix}$$

Solutions $x_1 = 2x_2 - 5x_3$

x_2, x_3 - free

All solutions $\vec{x}$ have the form

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2x_2 - 5x_3 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2x_2 \\ x_2 \\ 0 \end{bmatrix} + \begin{bmatrix} -5x_3 \\ 0 \\ x_3 \end{bmatrix}$$

$$= x_2 \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -5 \\ 0 \\ 1 \end{bmatrix}$$

The solution set is

$$\mathrm{Span}\left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -5 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

Parametric Vector Form of a Solution Set

Example (b) had a solution set consisting of vectors of the form $\mathbf{x} = x_3\mathbf{u}$. Example (c)'s solution set consisted of vector that look like $\mathbf{x} = x_2\mathbf{u} + x_3\mathbf{v}$. Since these are **linear combinations**, we could write the solution sets like

$$\text{Span}\{\mathbf{u}\} \quad \text{or} \quad \text{Span}\{\mathbf{u}, \mathbf{v}\}.$$

Instead of using the variables x_2 and/or x_3 we often substitute **parameters** such as s or t.
The forms

$$\mathbf{x} = s\mathbf{u}, \quad \text{or} \quad \mathbf{x} = s\mathbf{u} + t\mathbf{v}$$

are called **parametric vector forms.**

Example

The **parametric vector form** of the solution set of
$x_1 - 2x_2 + 5x_3 = 0$ is

$$\mathbf{x} = s\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -5 \\ 0 \\ 1 \end{bmatrix}, \quad \text{where } s, t \in \mathbb{R}.$$

Question: What geometric object is that solution set?

It's a plane in $\mathbb{R}^3$ containing $(2,1,0)$, $(-5,0,1)$
and the origin.

Nonhomogeneous Systems

Find all solutions of the nonhomogeneous system of equations

$$\begin{array}{rcrcrcr} 3x_1 & + & 5x_2 & - & 4x_3 & = & 7 \\ -3x_1 & - & 2x_2 & + & 4x_3 & = & -1 \\ 6x_1 & + & x_2 & - & 8x_3 & = & -4 \end{array}$$

Using technology

$$\text{rref}\left(\begin{bmatrix} 3 & 5 & -4 & 7 \\ -3 & -2 & 4 & -1 \\ 6 & 1 & -8 & -4 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 & -\frac{4}{3} & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

$$\left.\begin{array}{l} x_1 - \frac{4}{3}x_3 = -1 \\ x_2 = 2 \end{array}\right\} \Rightarrow \begin{array}{l} x_1 = -1 + \frac{4}{3}x_3 \\ x_2 = 2 \\ x_3 - \text{free} \end{array}$$

The solutions are

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 + \frac{4}{3}x_3 \\ 2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} + \begin{bmatrix} 4/3\, x_3 \\ 0 \\ x_3 \end{bmatrix}$$

$$= \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 4/3 \\ 0 \\ 1 \end{bmatrix}$$

This is the line $\text{span}\left\{\begin{bmatrix} 4/3 \\ 0 \\ 1 \end{bmatrix}\right\}$ translated to pass through $\vec{p} = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}$.

Solutions of Nonhomogeneous Systems

Note that the solution in this example has the form

$$\mathbf{x} = \mathbf{p} + t\mathbf{v}$$

with $\mathbf{p}$ and $\mathbf{v}$ fixed vectors and t a varying parameter. Also note that the $t\mathbf{v}$ part is the solution to the previous example with the right hand side all zeros. This is no coincidence!

$\mathbf{p}$ is called a **particular solution**, and $t\mathbf{v}$ is called a solution to the associated homogeneous equation.

Theorem

Suppose the equation $A\mathbf{x} = \mathbf{b}$ is consistent for a given $\mathbf{b}$. Let $\mathbf{p}$ be a solution. Then the solution set of $A\mathbf{x} = \mathbf{b}$ is the set of all vectors of the form

$$\mathbf{x} = \mathbf{p} + \mathbf{v}_h,$$

where $\mathbf{v}_h$ is any solution of the associated homogeneous equation $A\mathbf{x} = \mathbf{0}$.

We can use a row reduction technique to get all parts of the solution in one process.

Example

Find the solution set of the following system. Express the solution set in parametric vector form.

$$\begin{array}{rcrcrcrcr} x_1 & + & x_2 & - & 2x_3 & + & 4x_4 & = & 1 \\ 2x_1 & + & 3x_2 & - & 6x_3 & + & 12x_4 & = & 4 \end{array}$$

$$\begin{bmatrix} 1 & 1 & -2 & 4 & 1 \\ 2 & 3 & -6 & 12 & 4 \end{bmatrix}$$

$-2R_1 + R_2 \to R_2$

$$\begin{array}{ccccc} -2 & -2 & 4 & -8 & -2 \\ 2 & 3 & -6 & 12 & 4 \end{array}$$

$$\begin{bmatrix} 1 & 1 & -2 & 4 & 1 \\ 0 & 1 & -2 & 4 & 2 \end{bmatrix}$$

$-R_2 + R_1 \to R_1$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & -1 \\ 0 & 1 & -2 & 4 & 2 \end{bmatrix}$$

$x_1 = -1$

$x_2 = 2 + 2x_3 - 4x_4$

x_3, x_4 - free

The solutions

$$\vec{x} = \begin{bmatrix} -1 \\ 2 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 2 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ -4 \\ 0 \\ 1 \end{bmatrix}$$

$$\vec{x} = \begin{bmatrix} -1 \\ 2 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} 0 \\ 2 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ -4 \\ 0 \\ 1 \end{bmatrix}, \quad s, t \text{ in } \mathbb{R}.$$

Section 1.7: Linear Independence

We already know that a homogeneous equation $A\mathbf{x} = \mathbf{0}$ can be thought of as an equation in the column vectors of the matrix $A = [\mathbf{a}_1\ \mathbf{a}_2\ \cdots\ \mathbf{a}_n]$ as

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots x_n\mathbf{a}_n = \mathbf{0}.$$

And, we know that at least one solution (the trivial one $x_1 = x_2 = \cdots = x_n = 0$) always exists.

Whether or not there is a nontrivial solution gives us a way to characterize the vectors $\mathbf{a}_1, \ldots, \mathbf{a}_n$.

Definition: Linear Dependence/Independence

An indexed set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_p\}$ in $\mathbb{R}^n$ is said to be **linearly independent** if the vector equation

$$x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \cdots x_p\mathbf{v}_p = \mathbf{0}$$

has only the trivial solution.

The set $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_p\}$ is said to be **linearly dependent** if there exists a set of weights $c_1, c_2, \dots, c_p$ *at least one of which is nonzero* such that

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots c_p\mathbf{v}_p = \mathbf{0}.$$

(i.e. Provided the homogeneous equation possesses a nontrivial solution.)

An equation $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots c_p\mathbf{v}_p = \mathbf{0}$, with at least one $c_i \neq 0$, is called a **linear dependence relation**.

Special Cases

A set with two vectors $\{\mathbf{v}_1, \mathbf{v}_2\}$ is linearly dependent if one is a scalar multiple of the other.

If they are linearly dependent, then there exists c_1, c_2 not both zero such that

$$c_1\vec{v}_1 + c_2\vec{v}_2 = \vec{0}$$

We can assume $c_1 \neq 0$ (else relabel $\vec{v}_1$ and $\vec{v}_2$)

$$c_1\vec{v}_1 = -c_2\vec{v}_2 \quad \Longrightarrow \quad \vec{v}_1 = \frac{-c_2}{c_1}\vec{v}_2 = k\vec{v}_2$$

where $k = \dfrac{-c_2}{c_1}$

Example

Determine if the set is linearly dependent or linearly independent.

(a) $\mathbf{v}_1 = \begin{bmatrix} 2 \\ 4 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} -1 \\ -2 \end{bmatrix}$ Dependent : $\vec{v}_1 = -2\vec{v}_2$

(b) $\mathbf{v}_1 = \begin{bmatrix} 2 \\ 4 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$ Lin. independent.