August 26 Math 1190 sec. 51 Fall 2016

Section 1.2: Limits of Functions Using Properties of Limits

Theorem: If $f(x) = A$ where A is a constant, then for any real number c

$$\lim_{x\to c} f(x) = \lim_{x\to c} A = A$$

Theorem: If $f(x) = x$, then for any real number c

$$\lim_{x\to c} f(x) = \lim_{x\to c} x = c$$

Additional Limit Law Theorems

Suppose

$$\lim_{x\to c} f(x) = L, \quad \lim_{x\to c} g(x) = M, \quad \text{and } k \text{ is constant.}$$

Theorem: (Sums) $\lim_{x\to c}(f(x)+g(x)) = L+M$

Theorem: (Differences) $\lim_{x\to c}(f(x)-g(x)) = L-M$

Theorem: (Constant Multiples) $\lim_{x\to c} kf(x) = kL$

Theorem: (Products) $\lim_{x\to c} f(x)g(x) = LM$

Additional Limit Law Theorems

Suppose $\lim_{x \to c} f(x) = L$ and n is a positive integer.

Theorem: (Power) $\lim_{x \to c} (f(x))^n = L^n$

Note in particular that this tells us that $\lim_{x \to c} x^n = c^n$.

Theorem: (Root) $\lim_{x \to c} \sqrt[n]{f(x)} = \sqrt[n]{L}$ (if this is defined)

Combining the sum, difference, constant multiple and power laws:
Theorem: If $P(x)$ is a polynomial, then

$$\lim_{x \to c} P(x) = P(c).$$

Question

(1) $\lim_{x \to 2}(3x^2 - 4x + 7) =$

(a) 7

(b) DNE

(c) -11

(d) 11

If

$P(x) = 3x^2 - 4x + 7$

$P(2) = 3(2)^2 - 4(2) + 7$

$= 12 - 8 + 7 = 11$

Question

(2) Suppose that we have determined that $\lim_{x \to 7} f(x) = 13$.

True or False: It is acceptable to write this as

$$" \lim_{x \to 7} = 13 "$$

this is like

$$" \sqrt{} = 2 "$$

False! $\lim_{x \to c}$ must be followed by an expression $f(x)$.

Additional Limit Law Theorems

Suppose $\lim_{x \to c} f(x) = L, \quad \lim_{x \to c} g(x) = M$ and $M \neq 0$

Theorem: (Quotient) $\lim_{x \to c} \dfrac{f(x)}{g(x)} = \dfrac{L}{M}$

Combined with our result for polynomials:

Theorem: If $R(x) = \frac{p(x)}{q(x)}$ is a rational function, and c is in the domain of R, then

$$\lim_{x \to c} R(x) = R(c).$$

Example

Evaluate $\lim_{x \to 2} \dfrac{x^2 + 5}{x^2 + x - 1}$

$R(x) = \dfrac{x^2+5}{x^2+x-1}$ is rational.

also $2^2+2-1 = 5 \neq 0$ so 2 is in the domain of R.

so $\lim_{x \to 2} \dfrac{x^2+5}{x^2+x-1} = \dfrac{2^2+5}{2^2+2-1} = \dfrac{9}{5}$

Evaluate $\lim_{x \to 1} \frac{\sqrt{x+1}}{x+5}$

nth root property

sum property

Note $\lim_{x \to 1} \sqrt{x+1} = \sqrt{\lim_{x \to 1} (x+1)} = \sqrt{\lim_{x \to 1} x + \lim_{x \to 1} 1}$

$= \sqrt{1+1} = \sqrt{2}$

$\lim_{x \to 1} (x+5) = 1+5 = 6$

So $\lim_{x \to 1} \frac{\sqrt{x+1}}{x+5} = \frac{\sqrt{2}}{6}$

Additional Techniques: When direct laws fail

Evaluate if possible $\lim_{x \to 2} \dfrac{x^2 - x - 2}{x^2 - 4}$

If $R(x) = \dfrac{x^2 - x - 2}{x^2 - 4}$ then 2 is <u>not</u> in the domain of R.

Since $x^2 - x - 2$ also goes to zero as $x \to 2$, there must be a common factor $(x-2)$.

We can use some algebra

$$\lim_{x \to 2} \frac{x^2 - x - 2}{x^2 - 4} = \lim_{x \to 2} \frac{\cancel{(x-2)}(x+1)}{\cancel{(x-2)}(x+2)}$$

$x \neq 2$ so $x-2$ cancels

$$= \lim_{x \to 2} \frac{x+1}{x+2} = \frac{3}{4}$$

Additional Techniques: When direct laws fail

Evaluate if possible $\displaystyle\lim_{x\to 1} \frac{\sqrt{x+3}-2}{x-1}$

Both numerator and denominator take the value zero if $x=1$. This does indicate that $x-1$ is a common "factor." We'll rationalize.

The conjugate of $\sqrt{x+3}-2$ is $\sqrt{x+3}+2$

$$\lim_{x\to 1} \frac{\sqrt{x+3}-2}{x-1} = \lim_{x\to 1} \left(\frac{\sqrt{x+3}-2}{x-1}\right)\cdot\left(\frac{\sqrt{x+3}+2}{\sqrt{x+3}+2}\right)$$

multiply by $\nearrow$ 1

$$= \lim_{x \to 1} \frac{\left(\sqrt{x+3}\right)^2 - 2\sqrt{x+3} + 2\sqrt{x+3} - 2^2}{(x-1)\left(\sqrt{x+3}+2\right)}$$

$$= \lim_{x \to 1} \frac{x+3-4}{(x-1)\left(\sqrt{x+3}+2\right)}$$

$$x-1 = 1(x-1)$$

$$= \lim_{x \to 1} \frac{\cancel{x-1}}{\cancel{(x-1)}\left(\sqrt{x+3}+2\right)} =$$

$$= \lim_{x \to 1} \frac{1}{\sqrt{x+3}+2} = \frac{1}{\sqrt{1+3}+2} = \frac{1}{2+2} = \frac{1}{4}$$

Question

Evaluate if possible $\lim_{x \to 2} \dfrac{x-2}{\sqrt{x}-\sqrt{2}}$

(a) $\frac{1}{\sqrt{2}}$

(b) $\sqrt{2}$

(c) DNE

(d) $2\sqrt{2}$

$$= \lim_{x \to 2} \left(\frac{x-2}{\sqrt{x}-\sqrt{2}} \right) \cdot \frac{\sqrt{x}+\sqrt{2}}{\sqrt{x}+\sqrt{2}}$$

$$= \lim_{x \to 2} \frac{(x-2)(\sqrt{x}+\sqrt{2})}{x-2}$$

$$= \lim_{x \to 2} (\sqrt{x}+\sqrt{2}) = \sqrt{2}+\sqrt{2} = 2\sqrt{2}$$

Observations

In limit taking, the form "$\frac{0}{0}$" sometimes appears. This is called an **indeterminate form.** Standard strategies are

(1) Try to factor the numerator and denominator to see if a common factor–$(x - c)$–can be cancelled.

(2) If dealing with roots, try rationalizing to reveal a common factor.

The form

$$\text{,,}\frac{\text{nonzero constant}}{0}\text{,,}$$

is not indeterminate. It is undefined. When it appears, the limit doesn't exist.

Example

Let $f(x) = x^3 + 2x$. Determine the difference quotient

$$\frac{f(x+h) - f(x)}{h} \quad \text{for} \quad h \neq 0.$$

Next, take the limit as $h \to 0$ of this difference quotient.

$$f(x) = x^3 + 2x$$

$$f(x+h) = (x+h)^3 + 2(x+h)$$

$$= x^3 + 3x^2h + 3xh^2 + h^3 + 2x + 2h$$

$$\frac{f(x+h) - f(x)}{h} = \frac{\cancel{x^3} + 3x^2h + 3xh^2 + h^3 + \cancel{2x} + 2h - (\cancel{x^3} + \cancel{2x})}{h}$$

$$= \frac{3x^2h + 3xh^2 + h^3 + 2h}{h}$$

$$= \frac{\cancel{h}\left(3x^2 + 3xh + h^2 + 2\right)}{\cancel{h}}$$

$$= 3x^2 + 3xh + h^2 + 2$$

$$\lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \left(3x^2 + 3xh + h^2 + 2\right) =$$

$$= 3x^2 + 3x \cdot 0 + 0^2 + 2$$

$$= 3x^2 + 2$$