August 26 Math 1190 sec. 52 Fall 2016

Section 1.2: Limits of Functions Using Properties of Limits

Theorem: If $f(x) = A$ where A is a constant, then for any real number c

$$\lim_{x \to c} f(x) = \lim_{x \to c} A = A$$

Theorem: If $f(x) = x$, then for any real number c

$$\lim_{x \to c} f(x) = \lim_{x \to c} x = c$$

Additional Limit Law Theorems

Suppose

$$\lim_{x\to c} f(x) = L, \quad \lim_{x\to c} g(x) = M, \quad \text{and } k \text{ is constant.}$$

Theorem: (Sums) $\lim_{x\to c}(f(x)+g(x)) = L+M$

Theorem: (Differences) $\lim_{x\to c}(f(x)-g(x)) = L-M$

Theorem: (Constant Multiples) $\lim_{x\to c} kf(x) = kL$

Theorem: (Products) $\lim_{x\to c} f(x)g(x) = LM$

Additional Limit Law Theorems

Suppose $\lim_{x \to c} f(x) = L$ and n is a positive integer.

Theorem: (Power) $\lim_{x \to c} (f(x))^n = L^n$

Note in particular that this tells us that $\lim_{x \to c} x^n = c^n$.

Theorem: (Root) $\lim_{x \to c} \sqrt[n]{f(x)} = \sqrt[n]{L}$ (if this is defined)

Combining the sum, difference, constant multiple and power laws:
Theorem: If $P(x)$ is a polynomial, then

$$\lim_{x \to c} P(x) = P(c).$$

Question

(1) $\lim_{x \to 2} (3x^2 - 4x + 7) =$

(a) 7

(b) DNE

(c) -11

(d) 11

If $P(x) = 3x^2 - 4x + 7$

$P(2) = 3(2)^2 - 4(2) + 7$

$= 12 - 8 + 7 = 11$

Question

(2) Suppose that we have determined that $\lim_{x \to 7} f(x) = 13$.

this is akin to "$\int = 2$"

True or False: It is acceptable to write this as

$$" \lim_{x \to 7} = 13 "$$

"$\lim_{x \to c}$" must be followed by a function/expression.

Never an equal sign.

Additional Limit Law Theorems

$$\text{Suppose} \quad \lim_{x \to c} f(x) = L, \quad \lim_{x \to c} g(x) = M \quad \text{and} \quad M \neq 0$$

Theorem: (Quotient) $\displaystyle \lim_{x \to c} \frac{f(x)}{g(x)} = \frac{L}{M}$

Combined with our result for polynomials:

Theorem: If $R(x) = \frac{p(x)}{q(x)}$ is a rational function, and c is in the domain of R, then

$$\lim_{x \to c} R(x) = R(c).$$

Example

Evaluate $\lim_{x\to 2} \dfrac{x^2+5}{x^2+x-1}$

If $R(x) = \dfrac{x^2+5}{x^2+x-1}$ then R is rational.

Note $2^2+2-1 = 5 \neq 0$, so 2 is in the domain of R.

so $\lim_{x\to 2} \dfrac{x^2+5}{x^2+x-1} = \dfrac{2^2+5}{2^2+2-1} = \dfrac{9}{5}$

Evaluate $\lim_{x \to 1} \frac{\sqrt{x+1}}{x+5}$

Note $\lim_{x \to 1} (x+5) = 1+5 = 6$

If $g(x) = x+5$

$\lim_{x \to 1} g(x) = 6 \neq 0$

Also $\lim_{x \to 1} \sqrt{x+1} = \sqrt{\lim_{x \to 1} (x+1)} = \sqrt{1+1} = \sqrt{2}$

nth root property

So $\lim_{x \to 1} \frac{\sqrt{x+1}}{x+5} = \frac{\sqrt{2}}{6}$

Additional Techniques: When direct laws fail

Evaluate if possible $\lim_{x \to 2} \frac{x^2 - x - 2}{x^2 - 4}$

If $R(x) = \frac{x^2 - x - 2}{x^2 - 4}$, 2 is not in the domain.

But both numerator and denominator go to zero.

So $x - 2$ is a factor of both.

$$\lim_{x \to 2} \frac{x^2 - x - 2}{x^2 - 4} = \lim_{x \to 2} \frac{\cancel{(x-2)}(x+1)}{\cancel{(x-2)}(x+2)}$$

$$= \lim_{x \to 2} \frac{x+1}{x+2} = \frac{2+1}{2+2} = \frac{3}{4}$$

Additional Techniques: When direct laws fail

Evaluate if possible $\lim_{x \to 1} \dfrac{\sqrt{x+3} - 2}{x - 1}$

Both numerator and denominator tend to zero.
So again $x-1$ is a "factor." We'll rationalize
to get the factor out.

$$\lim_{x \to 1} \frac{\sqrt{x+3} - 2}{x-1} = \lim_{x \to 1} \left(\frac{\sqrt{x+3} - 2}{x-1} \right) \cdot \left(\frac{\sqrt{x+3} + 2}{\sqrt{x+3} + 2} \right)$$

multiply by 1

conjugate of $\sqrt{x+3} - 2$

$$= \lim_{x \to 1} \frac{\left(\sqrt{x+3}\right)^2 - \cancel{2\sqrt{x+3}} + \cancel{2\sqrt{x+3}} - 2^2}{(x-1)\left(\sqrt{x+3}+2\right)}$$

$$= \lim_{x \to 1} \frac{x+3-4}{(x-1)\left(\sqrt{x+3}+2\right)}$$

$$= \lim_{x \to 1} \frac{\cancel{x-1}}{\cancel{(x-1)}\left(\sqrt{x+3}+2\right)}$$

$$= \lim_{x \to 1} \frac{1}{\sqrt{x+3}+2} = \frac{1}{\sqrt{4}+2} = \frac{1}{4}$$

Question

Evaluate if possible $\lim\limits_{x \to 2} \dfrac{x-2}{\sqrt{x}-\sqrt{2}}$

(a) $\frac{1}{\sqrt{2}}$

(b) $\sqrt{2}$

(c) DNE

(d) $2\sqrt{2}$

$$= \lim_{x \to 2} \left(\frac{x-2}{\sqrt{x}-\sqrt{2}} \right) \cdot \left(\frac{\sqrt{x}+\sqrt{2}}{\sqrt{x}+\sqrt{2}} \right)$$

$$= \lim_{x \to 2} \frac{\cancel{(x-2)}(\sqrt{x}+\sqrt{2})}{\cancel{x-2}}$$

$$= \lim_{x \to 2} \left(\sqrt{x}+\sqrt{2} \right) = \sqrt{2}+\sqrt{2} = 2\sqrt{2}$$

Observations

In limit taking, the form "$\frac{0}{0}$" sometimes appears. This is called an **indeterminate form**. Standard strategies are

(1) Try to factor the numerator and denominator to see if a common factor–$(x-c)$–can be cancelled.

(2) If dealing with roots, try rationalizing to reveal a common factor.

The form

$$\text{,,}\frac{\text{nonzero constant}}{0}\text{,,}$$

is not indeterminate. It is undefined. When it appears, the limit doesn't exist.

Example

Let $f(x) = x^3 + 2x$. Determine the difference quotient

$$\frac{f(x+h) - f(x)}{h} \quad \text{for} \quad h \neq 0.$$

Next, take the limit as $h \to 0$ of this difference quotient.

$$f(x) = x^3 + 2x$$

$$f(x+h) = (x+h)^3 + 2(x+h)$$

$$= x^3 + 3x^2h + 3xh^2 + h^3 + 2x + 2h$$

$$\frac{f(x+h) - f(x)}{h} = \frac{\cancel{x^3} + 3x^2h + 3h^2x + h^3 + \cancel{2x} + 2h - (\cancel{x^3} + \cancel{2x})}{h}$$

$$= \frac{3x^2h + 3h^2x + h^3 + 2h}{h}$$

$$= \frac{\cancel{h}(3x^2 + 3hx + h^2 + 2)}{\cancel{h}}$$

$$= 3x^2 + 3hx + h^2 + 2$$

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} (3x^2 + 3hx + h^2 + 2)$$

$$= 3x^2 + 3x \cdot 0 + 0^2 + 2$$

$$= 3x^2 + 2$$

Section 1.3: Continuity

We have seen that their may or may not be a relationship between the quantities

$$\lim_{x \to c} f(x) \quad \text{and} \quad f(c).$$

One or the other (or both) may fail to exist. And even if both exist, they need not be equivalent.

We've also seen that for polynomials at least, that the limit at a point is the same as the function value at that point. Here, we explore this property that polynomials (and lots of other functions, but not all) share.

Definition: Continuity at a Point

Definition: A function f is continuous at a number c if

$$\lim_{x \to c} f(x) = f(c).$$

Note that three properties are contained in this statement:
(1) $f(c)$ is defined (i.e. c is in the domain of f),

(2) $\lim_{x \to c} f(x)$ exists, and

(3) the limit actually equals the function value.

If a function f is not continuous at c, we may say that f is **discontinuous** at c

Polynomials and Rational Functions

In the previous section, we saw that:

If P is any polynomial and c is any real number, then $\lim_{x \to c} P(x) = P(c)$, and

If R is any rational function and c is any number in the domain of R, then $\lim_{x \to c} R(x) = R(c)$.

Conclusion Theorem: Every rational function* is continuous at each number in its domain.

*Note that polynomials can be lumped in to the set of all rational functions.