August 26 Math 2306 sec 51 Fall 2015

Section 2.2: Separation of Variables

Recall that a first order ODE is called separable if it can be expressed in the form

$$\frac{dy}{dx} = g(x)h(y).$$

Assuming it's safe to divide by $h(y)$, we set $p(y) = 1/h(y)$. We solve (usually find an implicit solution) by **separating the variables.**

$$\frac{dy}{dx} = g(x)h(y) \quad \Longrightarrow \quad \int p(y)\,dy = \int g(x)\,dx$$

Giving a one parameter family of solutions defined implicitly by

$$P(y) = G(x) + C \quad \text{with } P \text{ and } G \text{ anti-derivatives of } p \text{ and } g.$$

Solve the ODE

$$\frac{dy}{dx} = -\frac{x}{y}$$

This is separable

$$\frac{dy}{dx} = -x\left(\frac{1}{y}\right) \Rightarrow y\frac{dy}{dx} = -x$$

$$y\frac{dy}{dx}dx = -x\,dx$$

$$y\,dy = -x\,dx \Rightarrow \int y\,dy = \int -x\,dx$$

$$\frac{y^2}{2} = \frac{-x^2}{2} + K$$

$$y^2 = -x^2 + C \qquad (C = 2k)$$

$$\Rightarrow \quad x^2 + y^2 = C$$

a one parameter family of

implicit solutions.

Solve the ODE

$$te^{t-y}\,dt - dy = 0 \quad \Rightarrow \quad -dy = -te^{t-y}\,dt$$

$$dy = te^{t}e^{-y}\,dt \qquad \text{divide by } e^{-y}$$

$$\frac{1}{e^{-y}}\,dy = te^{t}\,dt$$

$$e^{y}\,dy = te^{t}\,dt \quad \Rightarrow \quad \int e^{y}\,dy = \int te^{t}\,dt$$

$$e^{y} = \int te^{t}\,dt \qquad \text{Integrate by parts}$$

$$e^y = te^t - \int e^t \, dt$$

$u = t \qquad du = dt$

$v = e^t \qquad dv = e^t \, dt$

$$e^y = te^t - e^t + C$$

$$e^y - te^t + e^t = C$$

An IVP[1]

Solve the initial value problem

$$\frac{dQ}{dt} = -2(Q-70), \quad Q(0) = 180$$

$$\frac{1}{Q-70}\frac{dQ}{dt} = -2 \quad \Rightarrow \quad \frac{1}{Q-70}\frac{dQ}{dt}\,dt = -2\,dt$$

$$\frac{1}{Q-70}\,dQ = -2\,dt$$

$$\int \frac{1}{Q-70}\,dQ = \int -2\,dt$$

[1] Recall IVP stands for *initial value problem*.

$$\ln|Q-70| = -2t + C$$

exponentiate

$$e^{\ln|Q-70|} = |Q-70| = e^{-2t+C} = e^{C} e^{-2t}$$

Let $A = e^{C}$ or $-e^{C}$ or 0

to get rid of abs. value bars

$$Q - 70 = Ae^{-2t} \Rightarrow Q = Ae^{-2t} + 70$$

So $Q = Ae^{-2t} + 70$ and $Q(0) = 180$

$$Q(0) = Ae^0 + 70 = 180$$

$$A + 70 = 180 \Rightarrow A = 180 - 70 = 110$$

The solution to the IVP is

$$Q = 110e^{-2t} + 70.$$

Caveat regarding division by $h(y)$.

Solve the IVP by separation of variables[2]

$$\frac{dy}{dx} = x\sqrt{y}, \quad y(0) = 0$$

Solve the DE

$$\frac{1}{\sqrt{y}}\frac{dy}{dx} = x \quad \Rightarrow \quad \frac{1}{\sqrt{y}}\frac{dy}{dx}\,dx = x\,dx$$

$$\int y^{-1/2}\,dy = \int x\,dx \quad \Rightarrow \quad \frac{y^{1/2}}{1/2} = \frac{x^2}{2} + C$$

$$y^{1/2} = \frac{1}{2}\left(\frac{x^2}{2} + C\right) = \frac{x^2}{4} + k \qquad k = \frac{1}{2}C$$

[2] Remember that one solution is $y(x) = 0$ (for all x).

$$\sqrt{y} = \frac{x^2}{4} + k \qquad y(0) = 0$$

$$\sqrt{0} = \frac{0^2}{4} + k \implies 0 = k$$

So $\sqrt{y} = \frac{x^2}{4} \implies y = \frac{x^4}{16}$

Solves the IVP.

There is another solution, $y(x) = 0$.

Note the family of solutions

$$\sqrt{y} = \frac{x^2}{4} + k$$

does not contain the solution $y = 0$.

We discarded the solution $y=0$ when we divided by $\sqrt{y}$.

Separation of Variables-Lost Solutions

For $\frac{dy}{dx} = g(x)h(y) \qquad y(x_0) = y_0$

If $h(y_0) = 0$, then the constant solution $y(x) = y_0$ solves the IVP.

Note that if $y(x) = y_0$ (constant)

and $h(y_0) = 0$ then

solves the DE $\left\{ \begin{array}{l} \dfrac{dy}{dx} = \dfrac{d}{dx} y_0 = 0 \quad \text{and} \quad g(x)h(y) = g(x)h(y_0) = 0 \\ \text{so} \\ \dfrac{dy}{dx} = g(x)h(x) \end{array} \right.$

solves the I.C.* $\left\{ \begin{array}{l} \text{and} \\ y(x_0) = y_0 \end{array} \right.$

* I.C. = Initial Condition

In our example $h(y) = \sqrt{y}$ and $y_0 = 0$

so sure enough $h(y_0) = h(0) = \sqrt{0} = 0$

Solutions Defined by Integrals

Recall (Fundamental Theorem of Calculus)

$$\frac{d}{dx}\int_{x_0}^{x} g(t)\,dt = g(x) \quad \text{and} \quad \int_{x_0}^{x} \frac{dy}{dt}\,dt = y(x) - y(x_0).$$

Use this to solve

$$\frac{dy}{dx} = g(x), \quad y(x_0) = y_0$$

$$\frac{dy}{dt} = g(t) \quad \Rightarrow \quad \frac{dy}{dt}\,dt = g(t)\,dt$$

$$\Rightarrow \quad \int_{x_0}^{x} \frac{dy}{dt}\,dt = \int_{x_0}^{x} g(t)\,dt$$

$$\Rightarrow \quad y(x) - y(x_0) = \int_{x_0}^{x} g(t)\,dt$$

$$\Rightarrow \quad y(x) - y_0 = \int_{x_0}^{x} g(t)\,dt$$

$$\boxed{\Rightarrow \quad y(x) = y_0 + \int_{x_0}^{x} g(t)\,dt}$$

This is the solution to the IVP

y solves the DE

Note $\dfrac{dy}{dx} = \dfrac{d}{dx}\left(y_0 + \int_{x_0}^{x} g(t)\,dt \right) = g(x)$ ✓

y solves the I.C.

and $y(x_0) = y_0 + \int_{x_0}^{x_0} g(t)\,dt = y_0 + 0$ $\quad y(x_0) = y_0$ ✓

Example: Express the solution of the IVP in terms of an integral.

$$\frac{dy}{dx} = \sin(x^2), \quad y(\sqrt{\pi}) = 1$$

here $g(x) = \sin(x^2)$ which is not integrable (in terms of elementary functions).

$x_0 = \sqrt{\pi}$ and $y_0 = 1$

The solution

$$\boxed{y(x) = 1 + \int_{\sqrt{\pi}}^{x} \sin(t^2)\, dt}$$