August 26 Math 2306 sec. 51 Fall 2019

Section 2: Initial Value Problems

An initial value problem consists of an ODE with additional conditions.

Solve the equation [1]

$$\frac{d^n y}{dx^n} = f(x, y, y', \ldots, y^{(n-1)}) \tag{1}$$

subject to the *initial conditions*

$$y(x_0) = y_0, \quad y'(x_0) = y_1, \quad \ldots, y^{(n-1)}(x_0) = y_{n-1}. \tag{2}$$

The problem (1)–(2) is called an *initial value problem* (IVP).

[1] on some interval I containing x_0.

IVPs

First order case:

$$\frac{dy}{dx} = f(x, y), \quad y(x_0) = y_0$$

A first order ODE with one initial condition.

Second order case:

$$\frac{d^2y}{dx^2} = f(x, y, y'), \quad y(x_0) = y_0, \quad y'(x_0) = y_1$$

A second order ODE with two initial conditions. For example, y'' is the acceleration of a particle, y_0 is it's initial position, and y_1 is it's initial velocity.

Example

Solve the second order initial value problem

$$\frac{d^2y}{dx^2} = e^x - 12x, \quad y(0) = 0, \quad y'(0) = -1$$

Get y'

$$\frac{dy}{dx} = \int \frac{d^2y}{dx^2}\,dx = \int (e^x - 12x)\,dx$$

$$\frac{dy}{dx} = e^x - 12\frac{x^2}{2} + C_1 = e^x - 6x^2 + C_1$$

$$y = \int \frac{dy}{dx}\,dx = \int (e^x - 6x^2 + C_1)\,dx$$

$$y = e^x - 6\frac{x^3}{3} + c_1 x + c_2$$

$$y = e^x - 2x^3 + c_1 x + c_2$$

Now, we apply the initial conditions,

$$y'(0) = -1 = e^0 - 6(0^2) + c_1 \Rightarrow -1 = 1 + c_1$$

$$c_1 = -2$$

$$y(0) = 0 = e^0 - 2(0^3) + c_1(0) + c_2 \Rightarrow 0 = 1 + c_2$$

$$c_2 = -1$$

The solution to the IVP is

$$y = e^x - 2x^3 - 2x - 1$$

Example

Given that $y = c_1 x + \frac{c_2}{x}$ is a 2-parameter family of solutions of $x^2 y'' + xy' - y = 0$, solve the IVP

$$x^2 y'' + xy' - y = 0, \quad y(1) = 1, \quad y'(1) = 3$$

We already have the solutions to the ODE.

$y = c_1 x + \frac{c_2}{x}$. Apply the initial conditions,

$y' = c_1 - \frac{c_2}{x^2}$

$y(1) = 1 = c_1(1) + \frac{c_2}{1} \Rightarrow c_1 + c_2 = 1$

$y'(1) = 3 = c_1 - \frac{c_2}{1^2} \Rightarrow c_1 - c_2 = 3$

Solve this system. Add the equations

$$\begin{aligned} c_1 + c_2 &= 1 \\ c_1 - c_2 &= 3 \\ \hline 2c_1 &= 4 \end{aligned} \Rightarrow c_1 = 2$$

Substitute into 1st equation

$$c_2 = 1 - c_1 = 1 - 2 = -1$$

The solution to the IVP is

$$\boxed{y = 2x - \frac{1}{x}}$$

Example

Consider the IVP

$$y' + 2xy = 0, \quad y(0) = y_0$$

where y_0 is some real number. Verify that $y = y_0 e^{-x^2}$ is a solution to this IVP.

We need to show that y solves the ODE and satisfies the initial condition (IC).

ODE: $y = y_0 e^{-x^2}$, $y' = y_0 e^{-x^2}(-2x) = -2xy_0 e^{-x^2}$

$$y' + 2xy \overset{?}{=} 0$$

$$-2xy_0 e^{-x^2} + 2x\left(y_0 e^{-x^2}\right) = 0$$

$$-2xy_0 e^{-x^2} + 2xy_0 e^{-x^2} = 0$$

$$0 = 0 \quad \checkmark$$

y solves the ODE.

I.C. $\quad y = y_0 e^{-x^2}, \quad y(0) = y_0 e^{-0^2} = y_0 \cdot 1 = y_0$

The function also satisfies the IC, so it satisfies the IVP.

Graphical Interpretation

The solutions are $y = Ce^{-x^2}$

Integral curves

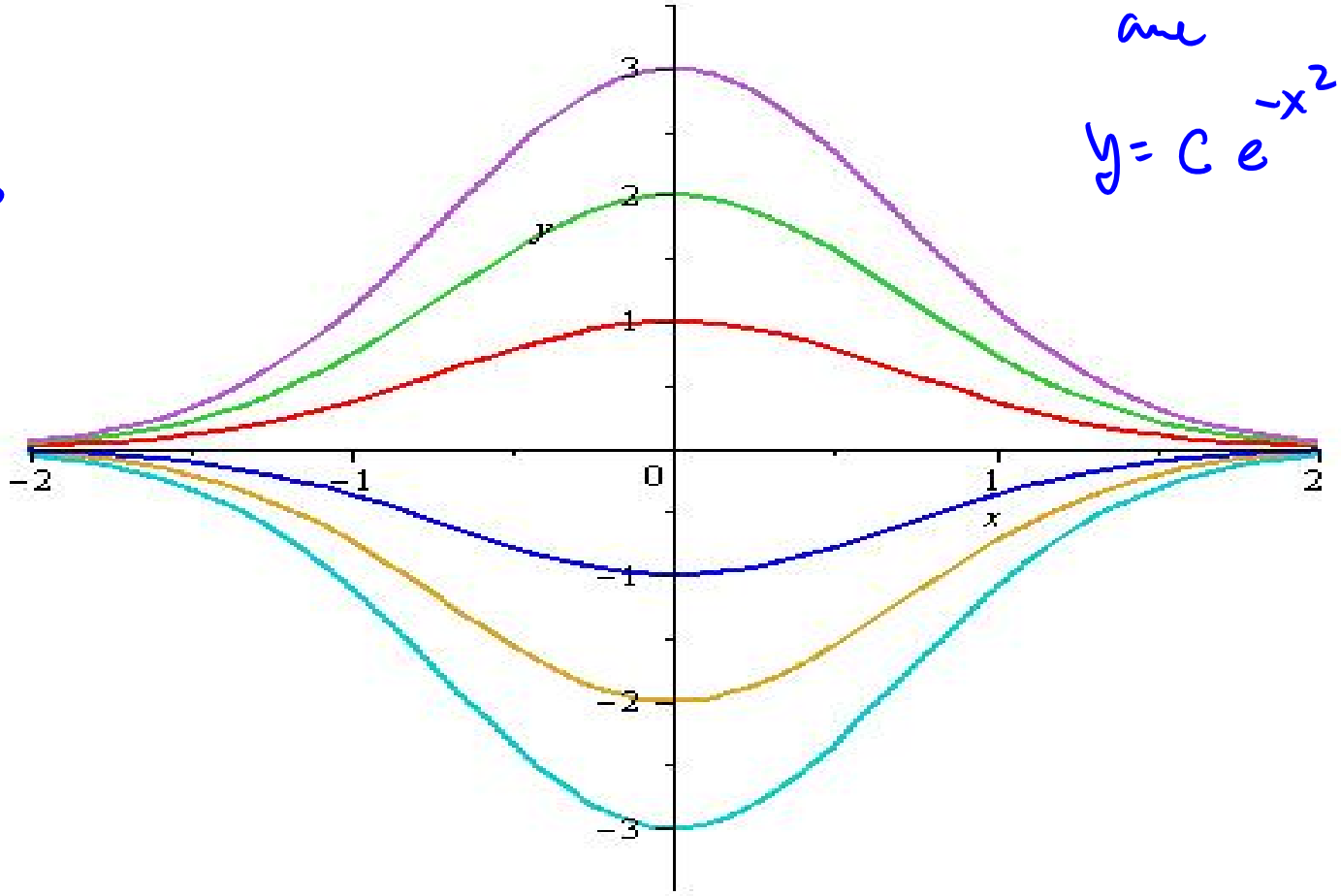

Figure: Each curve solves $y' + 2xy = 0$, $y(0) = y_0$. Each colored curve corresponds to a different value of y_0

Existence and Uniqueness

Two important questions we can always pose (and sometimes answer) are

(1) Does an IVP have a solution? (existence) and

(2) If it does, is there just one? (uniqueness)

Hopefully it's obvious that we can't solve $\left(\frac{dy}{dx}\right)^2 + 1 = -y^2$.

Left side ≥ 1

Right side ≤ 0

Uniqueness

Consider the IVP

$$\frac{dy}{dx} = x\sqrt{y} \quad y(0) = 0$$

Verify that $y = \frac{x^4}{16}$ is a solution of the IVP. And find a second solution of the IVP by clever guessing.

ODE: $y = \frac{x^4}{16}$, $y' = \frac{4x^3}{16} = \frac{x^3}{4}$

and $x\sqrt{y} = x\sqrt{\frac{x^4}{16}} = x\left|\frac{x^2}{4}\right| = \frac{x^3}{4}$

so $y' = \frac{x^3}{4} = x\sqrt{y}$

and $y(0) = \frac{0^4}{16} = 0$ y solves the IC too.

$y(x) = 0$ is another solution to the IVP (called the trivial solution).