August 26 Math 2306 sec 54 Fall 2015

Section 2.2: Separation of Variables

Definition: The first order equation $y' = f(x,y)$ is said to be **separable** if the right side has the form

$$f(x,y) = g(x)h(y).$$

That is, a separable equation is one that has the form

$$\frac{dy}{dx} = g(x)h(y).$$

The right hand side must be able to be expressed as the **product** of a function of only x and a function of only y.

Solving Separable Equations

Recall that from $\frac{dy}{dx} = g(x)$, we can integrate both sides

$$\int \frac{dy}{dx}\,dx = \int g(x)\,dx.$$

$$y = G(x) + C$$

where $G(x)$ is any antiderivative of $g(x)$

We'll use this observation!

Solving Separable Equations

Let's assume that it's safe to divide by $h(y)$ and let's set $p(y) = 1/h(y)$. We solve (usually find an implicit solution) by **separating the variables**.

$$\frac{dy}{dx} = g(x)h(y) \quad \Rightarrow \quad \frac{1}{h(y)}\frac{dy}{dx} = g(x)$$

$$\Rightarrow \quad \frac{1}{h(y)}\frac{dy}{dx}\,dx = g(x)\,dx$$

$$p(y)\,dy = g(x)\,dx$$

$$\int p(y)\,dy = \int g(x)\,dx$$

$$P(y) = G(x) + C$$

where P and G are antiderivatives of p and g, respectively.

$$P(y) = G(x) + C$$

is a one parameter family of solutions given implicitly.

Solve the ODE

$$\frac{dy}{dx} = -\frac{x}{y}$$

$$\frac{dy}{dx} = -x\left(\frac{1}{y}\right)$$

$$y\,\frac{dy}{dx} = -x \quad \Rightarrow \quad y\,\frac{dy}{dx}\,dx = -x\,dx$$

$$y\,dy = -x\,dx$$

$$\int y\,dy = \int -x\,dx$$

$$\frac{y^2}{2} = -\frac{x^2}{2} + K$$

$$y^2 = -x^2 + C \qquad \text{where} \quad C = 2k$$

$$x^2 + y^2 = C$$

One parameter family of implicit solutions.

Solve the ODE

$$te^{t-y}\,dt - dy = 0 \quad \Longrightarrow \quad -dy = -te^{t-y}\,dt$$

$$dy = te^{t} \cdot e^{-y}\,dt$$

$$\frac{1}{e^{-y}}\,dy = te^{t}\,dt$$

$$e^{y}\,dy = te^{t}\,dt \Rightarrow \int e^{y}\,dy = \int te^{t}\,dt$$

$$e^{y} = \int te^{t}\,dt$$

Integrate by parts

$$e^y = te^t - \int e^t\,dt \qquad u = t \quad du = dt$$

$$v = e^t \quad dv = e^t\,dt$$

$$e^y = te^t - e^t + C$$

a one parameter family of implicit solutions.

An IVP[1]

Solve the initial value problem

$$\frac{dQ}{dt} = -2(Q-70), \quad Q(0) = 180$$

Solve the DE first

$$\frac{1}{Q-70}\frac{dQ}{dt} = -2 \Rightarrow \frac{1}{Q-70}\frac{dQ}{dt}\,dt = -2\,dt$$

$$\frac{1}{Q-70}\,dQ = -2\,dt \Rightarrow \int \frac{1}{Q-70}\,dQ = \int -2\,dt$$

[1] Recall IVP stands for *initial value problem*.

$$\ln|Q-70| = -2t + C$$

I'll solve for Q before using the I.C. (initial condition)

Exponentiate

$$e^{\ln|Q-70|} = e^{-2t+C}$$

$$|Q-70| = e^{C} e^{-2t}$$

Let $A = e^{C}$ or $-e^{C}$ or 0

then we get rid of abs. value bars

$$Q - 70 = Ae^{-2t}$$

$$Q = Ae^{-2t} + 70$$

$Q(0) = 180$

impose the I.C.

$$Q(0) = Ae^{0} + 70 = 180$$

$$A + 70 = 180 \Rightarrow A = 180 - 70 = 110$$

The solution to the IVP is

$$\boxed{Q = 110e^{-2t} + 70.}$$