August 24 Math 2306 sec. 53 Fall 2018

Section 4: First Order Equations: Linear

- Put the equation in standard form $y' + P(x)y = f(x)$, and correctly identify the function $P(x)$.
- Obtain the integrating factor $\mu(x) = \exp\left(\int P(x)\,dx\right)$.
- Multiply both sides of the equation (in standard form) by the integrating factor μ. The left hand side **will always** collapse into the derivative of a product

$$\frac{d}{dx}[\mu(x)y] = \mu(x)f(x).$$

- Integrate both sides, and solve for y.

$$y(x) = \frac{1}{\mu(x)}\int \mu(x)f(x)\,dx = e^{-\int P(x)\,dx}\left(\int e^{\int P(x)\,dx} f(x)\,dx + C\right)$$

Solve the IVP

$$x\frac{dy}{dx} - y = 2x^2, \; x > 0 \quad y(1) = 5$$

Put in standard form
(divide by x)

$$\frac{dy}{dx} - \frac{1}{x}\, y = 2x \qquad P(x) = \frac{-1}{x}$$

Integrating factor $\quad \mu = e^{\int P(x)\,dx} = e^{\int \frac{-1}{x}\,dx} = e^{-\ln x} = e^{\ln x^{-1}} = x^{-1}$

So $\quad \mu = x^{-1} \qquad$ Mult. by μ

$$x^{-1}\left(\frac{dy}{dx} - \frac{1}{x}\, y\right) = x^{-1}\,(2x)$$

$$\frac{d}{dx}\left[x^{-1} y\right] = 2$$

$$\int \frac{d}{dx}\left[x^{-1}y\right]dx = \int 2\,dx$$

$$x^{-1}y = 2x + C$$

$$y = 2x^2 + Cx$$

a one parameter family of solutions to the ODE

Now use $y(1) = 5$.

$$5 = 2(1)^2 + C(1) = 2 + C \implies C = 3$$

The solution to the IVP is

$$y = 2x^2 + 3x$$

The family of solutions

was $y = 2x^2 + Cx$

y_p

particular solution

y_c

complementary solution

Verify

Just for giggles, lets verify that our solution $y = 2x^2 + 3x$ really does solve the differential equation we started with

$$x\frac{dy}{dx} - y = 2x^2.$$

$y = 2x^2 + 3x$ so $y' = 4x + 3$

$$x\frac{dy}{dx} - y \stackrel{?}{=} 2x^2$$

$$x(4x+3) - (2x^2+3x) =$$

$$4x^2 + 3x - 2x^2 - 3x =$$

$$4x^2 - 2x^2 =$$

yes, it is a solution

$$2x^2 = 2x^2$$

Steady and Transient States

For some linear equations, the term y_c decays as x (or t) grows. For example

$$\frac{dy}{dx} + y = 3xe^{-x} \quad \text{has solution} \quad y = \frac{3}{2}x^2e^{-x} + Ce^{-x}.$$

$$\text{Here,} \quad y_p = \frac{3}{2}x^2e^{-x} \quad \text{and} \quad y_c = Ce^{-x}.$$

Such a decaying complementary solution is called a **transient state.**

The corresponding particular solution is called a **steady state.**