August 28 Math 2306 sec 51 Fall 2015

Section 2.3: First Order Linear Equations

A first order linear equation has the form

$$a_1(x)\frac{dy}{dx} + a_0(x)y = g(x).$$

If $g(x) = 0$ the equation is called **homogeneous.** Otherwise it is called **nonhomogeneous.**

Provided $a_1(x) \neq 0$ on the interval I of definition of a solution, we can write the **standard form** of the equation

$P(x) = a_0(x)/a_1(x)$

$$\frac{dy}{dx} + P(x)y = f(x).$$

$f(x) = g(x)/a_1(x)$

We'll be interested in equations (and intervals I) for which P and f are continuous on I.

Solutions (the General Solution)

$$\frac{dy}{dx} + P(x)y = f(x).$$

It turns out the solution will always have a basic form of $y = y_c + y_p$ where

- y_c is called the **complementary** solution and would solve the problem
$$\frac{dy}{dx} + P(x)y = 0$$
(called the associated homogeneous equation), and
- y_p is called the **particular** solution, and is heavily influenced by the function $f(x)$.

The cool thing is that our solution method will get both parts in one process—we won't get this benefit with higher order equations!

Motivating Example

$$x^2\frac{dy}{dx}+2xy=e^x$$

The left side is the product rule

$$\frac{d}{dx}\left(x^2y\right)=x^2y'+2xy$$

So our equation is

$$\frac{d}{dx}\left(x^2y\right)=e^x$$

we want to find y

Start by taking an anti derivative

$$\int\frac{d}{dx}\left(x^2y\right)dx=\int e^x\,dx$$

$$x^2 y = e^x + C$$

$$\Rightarrow \quad y(x) = \frac{e^x}{x^2} + \frac{C}{x^2}$$

(on an interval for which $x^2 \neq 0$)

Derivation of Solution via Integrating Factor

Solve the equation in standard form

$$\frac{dy}{dx} + P(x)y = f(x)$$

The idea here is to multiply both sides of this equation by some function $\mu(x)$ so that the resulting left side is the product rule

$$\frac{d}{dx}\left(\mu(x)y\right) = \mu y' + \mu' y$$

Multiply the eqn by μ: $\quad \mu y' + \mu P(x) y = \mu f(x)$

We require

$$\mu \cancel{y'} + \mu' y = \mu \cancel{y'} + \mu P y \implies$$

$$\mu' y = \mu P y \implies$$

$$\mu' = \mu P \quad \text{i.e.} \quad \frac{d\mu}{dx} = \mu P(x)$$

This is first order separable.

$$\frac{1}{\mu} \frac{d\mu}{dx} dx = P(x)\, dx$$

$$\int \frac{1}{\mu} d\mu = \int P(x)\, dx$$

$$\Rightarrow \quad \ln|\mu| = \int P(x)\,dx \qquad \text{exponentiate}$$

$$|\mu| = e^{\int P(x)\,dx} \qquad \text{(we'll drop the abs. bars)}$$

$$\boxed{\mu(x) = e^{\int P(x)\,dx}}$$

Note

$$\mu' = e^{\int P(x)\,dx} \cdot P(x)$$

$$= \mu P(x)$$

$$\frac{d}{dx}\left[\mu(x)\, y\right] = \mu y' + \mu' y$$

$$= \mu y' + \mu P(x)\, y$$

$\mu(x) = e^{\int P(x)\,dx}$ is an integrating factor

From $y' + P(x)y = f(x)$ mult. by μ

$\mu y' + \mu P y = \mu f \Rightarrow$

$\frac{d}{dx}[\mu y] = \mu f$ Integrate

$\int \frac{d}{dx}(\mu y)\,dx = \int \mu(x) f(x)\,dx$

This is the general solution

$\mu y = \int \mu(x) f(x)\,dx + C \Rightarrow \boxed{y = \mu^{-1} \int \mu(x) f(x)\,dx + C\mu^{-1}}$

General Solution of First Order Linear ODE

- Put the equation in standard form $y' + P(x)y = f(x)$, and correctly identify the function $P(x)$.
- Obtain the integrating factor $\mu(x) = \exp\left(\int P(x)\,dx\right)$.
- Multiply both sides of the equation (in standard form) by the integrating factor μ. The left hand side **will always** collapse into the derivative of a product

$$\frac{d}{dx}[\mu(x)y] = \mu(x)f(x).$$

- Integrate both sides, and solve for y.

$$y(x) = \frac{1}{\mu(x)}\int \mu(x)f(x)\,dx = e^{-\int P(x)\,dx}\left(\int e^{\int P(x)\,dx} f(x)\,dx + C\right)$$