August 28 Math 2306 sec 51 Fall 2015

Section 2.3: First Order Linear Equations

A first order linear equation has the form

$$a_1(x)\frac{dy}{dx}+a_0(x)y=g(x).$$

If g(x) = 0 the equation is called **homogeneous**. Otherwise it is called **nonhomogeneous**.

Provided $a_1(x) \neq 0$ on the interval I of definition of a solution, we can write the **standard form** of the equation $\Pr_{(x) : A_0(x)} / A_{A_0(x)}$

$$\frac{dy}{dx} + P(x)y = f(x). \qquad f(x) = \frac{g(x)}{a} / a_{1}(x)$$

We'll be interested in equations (and intervals I) for which P and f are continuous on I.

Solutions (the General Solution)

$$\frac{dy}{dx} + P(x)y = f(x).$$

It turns out the solution will always have a basic form of $y = y_c + y_p$ where

▶ y_c is called the **complementary** solution and would solve the problem

$$\frac{dy}{dx} + P(x)y = 0$$

(called the associated homogeneous equation), and

 \triangleright y_p is called the **particular** solution, and is heavily influenced by the function f(x).

The cool thing is that our solution method will get both parts in one process—we won't get this benefit with higher order equations!



Motivating Example

$$x^2\frac{dy}{dx} + 2xy = e^x$$

The left side is the product rule $\frac{d}{dx}(x^2y) = x^2y^1 + 2xy$

So our equation is

$$\frac{d}{dx}(x^2b) = e^x$$
 we want to find y

Start by taking on anti-derivative

 $\int \frac{d}{dx}(x^2b) dx = \int e^x dx$

$$x^{2}y = e^{x} + C$$

$$\Rightarrow y(x) = \frac{e^{x}}{x^{2}} + \frac{C}{x^{2}} \qquad (e^{x} e^{x} + C)$$

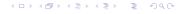
Derivation of Solution via Integrating Factor

Solve the equation in standard form

$$\frac{dy}{dx} + P(x)y = f(x)$$

The idea here is to meltiply both sides of this equation by some function $\mu(x)$ so that the resulting left side is the product rule $d(\mu(x)) = \mu(x) + \mu'(x)$

Mustiply the ego by p: py + pP(x) y= pof(x)



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We require

$$\mu g' + \mu' g = \mu g' + \mu P g \Rightarrow$$

$$\mu' g = \mu P g \Rightarrow$$

$$\mu' g \Rightarrow$$

$$\mu$$

This is first order separable.

$$\int \frac{dx}{dx} dx = P(x) dx$$

$$\int \frac{dx}{dx} dx = \int P(x) dx$$

Infal = I P(x) dx exponenticle (we'll drop the) abs. bers) m= frxxx f(x) = e f(x) dx Note spendx n'= e . P(x) = p. P(x) = my' + mP(x) &

$$\mu(x) = e^{\int P(x) dx}$$
 is an integrating factor

$$\int \frac{d}{dx} \left[\mu y \right] = \mu f \qquad \text{Integrate}$$

$$\int \frac{d}{dx} \left(\mu y \right) dx = \int \mu(x) f(x) dx \qquad \text{This general}$$

$$\mu y = \int \mu(x) f(x) dx + C \Rightarrow y = \mu \int \mu(x) f(x) dx + C \mu$$
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General Solution of First Order Linear ODE

- ▶ Put the equation in standard form y' + P(x)y = f(x), and correctly identify the function P(x).
- ▶ Obtain the integrating factor $\mu(x) = \exp(\int P(x) dx)$.
- Multiply both sides of the equation (in standard form) by the integrating factor μ . The left hand side **will always** collapse into the derivative of a product

$$\frac{d}{dx}[\mu(x)y] = \mu(x)f(x).$$

Integrate both sides, and solve for y.

$$y(x) = \frac{1}{\mu(x)} \int \mu(x) f(x) \, dx = e^{-\int P(x) \, dx} \left(\int e^{\int P(x) \, dx} f(x) \, dx + C \right)$$

