August 28 Math 2306 sec 54 Fall 2015

Section 2.2: Separation of Variables

We solved a separable equation $y' = g(x)h(y)$ by dividing through by h to *separate* the variables.

$$\int \frac{1}{h(y)}\,dy = \int g(x)\,dx$$

This required us to assume that we could divide by h—i.e. that $h(y) \neq 0$ on the interval of interest.

Caveat regarding division by $h(y)$.

Solve the IVP by separation of variables[1]

$$\frac{dy}{dx} = x\sqrt{y}, \quad y(0) = 0$$

$$\frac{dy}{dx} = x\, y^{1/2} \quad \Rightarrow$$

$$\Rightarrow \quad \frac{1}{y^{1/2}} \frac{dy}{dx} = x \qquad \Rightarrow \quad y^{-1/2} \frac{dy}{dx}\, dx = x\, dx$$

$$\int y^{-1/2}\, dy = \int x\, dx \qquad \Rightarrow \quad \frac{y^{1/2}}{1/2} = \frac{x^2}{2} + C$$

$$\Rightarrow \quad y^{1/2} = \frac{x^2}{4} + \frac{1}{2} C$$

[1] Remember that one solution is $y(x) = 0$ (for all x).

$$y^{1/2} = \frac{x^2}{4} + k \qquad \left(k = \tfrac{1}{2}C\right)$$

$$y = \left(\frac{x^2}{4} + k\right)^2$$

use $y(0)=0$ to get $\quad y(0) = 0 = \left(\frac{0^2}{4} + k\right)^2 = k^2$

$\Rightarrow \quad k = 0 \quad$ and $\quad y = \left(\frac{x^2}{4}\right)^2 = \frac{x^4}{16}$

we get the one solution

$$y = \frac{x^4}{16}$$

The family of solutions from separating the variables is

$$y = \left(\frac{x^2}{4} + k\right)^2$$

The solution $y(x) = 0$ is <u>not</u> in this family.

We lost the solution when we divided by $\sqrt{y}$.

Losing a Solution

If $h(y_0) = 0$, the the constant function $y(x) = y_0$ solves the IVP

$$\frac{dy}{dx} = g(x)h(y), \quad y(x_0) = y_0.$$

When separating the variables, we may inadvertently discard this solution.

In our previous example, $h(y) = \sqrt{y}$ and $y_0 = 0$ so that $h(y_0) = \sqrt{0} = 0$.

Solutions Defined by Integrals

Recall (Fundamental Theorem of Calculus)

$$\frac{d}{dx}\int_{x_0}^{x} g(t)\,dt = g(x) \quad \text{and} \quad \int_{x_0}^{x} \frac{dy}{dt}\,dt = y(x) - y(x_0).$$

Use this to solve

$$\frac{dy}{dx} = g(x), \quad y(x_0) = y_0$$

$$\frac{dy}{dt} = g(t) \quad \Rightarrow \quad \int_{x_0}^{x} \frac{dy}{dt}\,dt = \int_{x_0}^{x} g(t)\,dt$$

$$\Rightarrow \quad y(t)\Big|_{x_0}^{x} = \int_{x_0}^{x} g(t)\,dt$$

$$y(x) - y(x_0) = \int_{x_0}^{x} g(t)\,dt$$

$$y(x) - y_0 = \int_{x_0}^{x} g(t)\,dt \Rrightarrow \boxed{y = y_0 + \int_{x_0}^{x} g(t)\,dt}$$

Verify:

$$\frac{dy}{dx} = \frac{d}{dx}\left(y_0 + \int_{x_0}^{x} g(t)\,dt\right) = 0 + \frac{d}{dx}\int_{x_0}^{x} g(t)\,dt = g(x)$$

$$y(x_0) = y_0 + \int_{x_0}^{x_0} g(t)\,dt = y_0 + 0 = y_0$$

Example: Express the solution of the IVP in terms of an integral.

$$\frac{dy}{dx} = \sin(x^2), \quad y(\sqrt{\pi}) = 1$$

Here $g(t) = \sin(t^2)$

$x_0 = \sqrt{\pi}$ and $y_0 = 1$

The solution is

$$y = 1 + \int_{\sqrt{\pi}}^{x} \sin(t^2)\, dt$$

Section 2.3: First Order Linear Equations

A first order linear equation has the form

$$a_1(x)\frac{dy}{dx} + a_0(x)y = g(x).$$

If $g(x) = 0$ the equation is called **homogeneous.** Otherwise it is called **nonhomogeneous.**

Provided $a_1(x) \neq 0$ on the interval I of definition of a solution, we can write the **standard form** of the equation

$$\frac{dy}{dx} + P(x)y = f(x).$$

$P(x) = a_0(x)/a_1(x)$

$f(x) = g(x)/a_1(x)$

We'll be interested in equations (and intervals I) for which P and f are continuous on I.

Solutions (the General Solution)

$$\frac{dy}{dx} + P(x)y = f(x).$$

It turns out the solution will always have a basic form of $y = y_c + y_p$ where

- y_c is called the **complementary** solution and would solve the problem
  $$\frac{dy}{dx} + P(x)y = 0$$
  (called the associated homogeneous equation), and
- y_p is called the **particular** solution, and is heavily influenced by the function $f(x)$.

The cool thing is that our solution method will get both parts in one process—we won't get this benefit with higher order equations!

Motivating Example

$$x^2\frac{dy}{dx}+2xy=e^x$$

The left side is the derivative (product rule)

$$\frac{d}{dx}\left(x^2y\right).$$

The equation is $\frac{d}{dx}\left(x^2y\right)=e^x$.

So

$$\frac{d}{dx}\left(x^2y\right)\,dx = e^x\,dx$$

$$\int \frac{d}{dx}\left(x^2y\right)\,dx = \int e^x\,dx$$

$$x^2 y = e^x + C$$

$$y = \frac{e^x}{x^2} + \frac{C}{x^2}$$

(assuming $x \neq 0$ on an interval on an interval of interest)

Derivation of Solution via Integrating Factor

Solve the equation in standard form

$$\frac{dy}{dx} + P(x)y = f(x)$$

Goal: multiply this equation by a function $\mu(x)$ such that the resulting left hand side becomes $\frac{d}{dx}\left(\mu(x)\, y\right)$.

We need to find μ

$$\frac{d}{dx}\left(\mu(x)\, y\right) = \mu y' + \mu' y$$

$$\mu y' + \mu P(x)\, y = \mu f(x)$$

we require $\quad \cancel{\mu y'} + \mu' y = \cancel{\mu y'} + \mu P y$

$$\Rightarrow \quad \mu' \cancel{y} = \mu P \cancel{y}$$

$$\Rightarrow \quad \mu' = \mu P$$ a separable equation

$$\frac{d\mu}{dx} = \mu P(x)$$ separate

$$\int \frac{1}{\mu}\, d\mu = \int P(x)\, dx$$

$$\ln|\mu| = \int p(x)\,dx + C$$

exponentiate

$$|\mu| = e^{\int p(x)\,dx + C} = A\,e^{\int p(x)\,dx}$$

let $A = 1$ or -1 to obtain

$$\mu = e^{\int p(x)\,dx}$$

This is called an integrating factor

We'll finish this construction on Monday.