August 29 MATH 1113 sec. 51 Fall 2018

Section 2.4: Symmetry

Consider the function $f(x) = 2x^2 + 1$. Suppose we wished to plot the new function $h(x) = f(-x)$.

$f(-x) \rightarrow$ reflection in y-axis

Find the formula for $h(x) = f(-x)$.

$$h(x) = f(-x) = 2(-x)^2 + 1 = 2(-x)(-x) + 1 = 2x^2 + 1$$

This is the same as $f(x)$.

Even Functions

For $f(x) = 2x^2 + 1$, we found that

$$f(-x) = 2(-x)^2 + 1 = 2x^2 + 1 = f(x).$$

Since the graph of $f(-x)$ is obtained from f by reflection in the y-axis, and $f(-x)$ and $f(x)$ are the same for each x, it must be that

the graph of f is its own reflection in the y-axis!

Definition: A function f is called an **even function** if

$$f(-x) = f(x)$$

for each x in its domain. We can say that such a function has **even symmetry**.

Even Symmetry

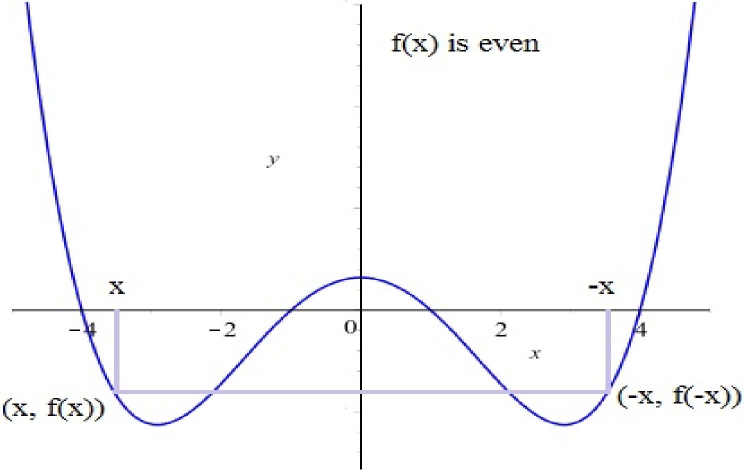

Figure: The graph to the left of the y-axis is the mirror image of the graph on the right side if a function is even.

Question

Suppose f is an even function whose graph passes through the point $(-2, 5)$. Which of the following points must be on the graph of f?

we know $f(-2) = 5$

since f is even

$f(2) = 5$

i.e. $f(-(-2)) = f(-2) = 5$

(a) $(5, -2)$

(b) $(-2, -5)$

(c) $(2, -5)$

(d) $(2, 5)$

(e) There isn't enough information to determine if any of the above **must** be on the graph.

Symmetry

Consider the function $f(x) = x - \frac{1}{2}x^3$, and let $g(x) = f(-x)$.

Find the formula for $g(x)$.

$$g(x) = f(-x) = (-x) - \frac{1}{2}(-x)^3$$

$$= -x - \frac{1}{2}(-x)(-x)(-x)$$

$$= -x - \frac{1}{2}(-x^3) = -x + \frac{1}{2}x^3$$

$$= -\left(x - \frac{1}{2}x^3\right) = -f(x)$$

↑ reflection in y-axis

↑ reflection in x-axis

Odd Functions

For $f(x) = x - \frac{1}{2}x^3$, we found that

$$f(-x) = (-x) - \frac{1}{2}(-x)^2 = -x + \frac{1}{2}x^3 = -\left(x - \frac{1}{2}x^3\right) = -f(x).$$

So $f(-x)$ is the reflection in the y-axis, and it's equal to the reflection in the x-axis. That is

the reflection of f in the y-axis is its reflection in the x-axis!

Definition: A function f is called an **odd function** if

$$f(-x) = -f(x)$$

for each x in its domain. We can say that such a function has **odd symmetry**.

Odd Symmetry

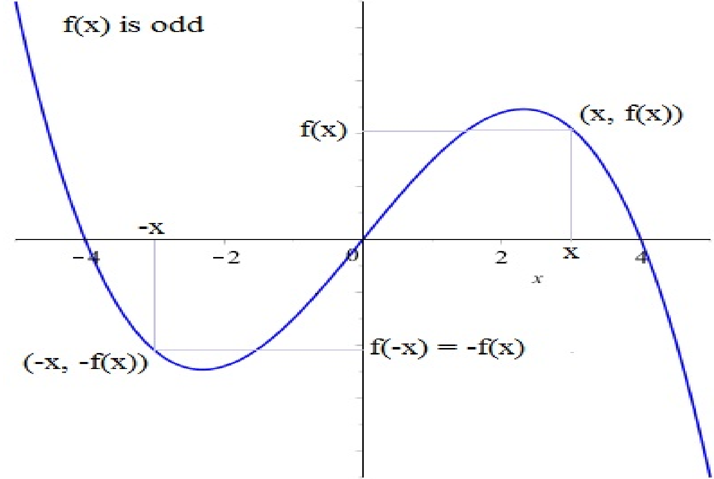

Figure: The graph of f to the left of the y-axis can be obtained by reflecting the graph on the right twice—through the y-axis and then the x-axis.

Even and Odd Symmetry

- Polynomials with only even powers (including 0) are even functions. Polynomials with only odd powers are odd functions.
- Even symmetry is called symmetry with respect to the y-axis.
- Odd symmetry is called symmetry with respect to the origin.
- Not all functions have symmetry (for example polynomials with both even and odd power terms). Some important functions have known symmetry.

Question

Suppose f is an **odd** function and that 0 is in the domain of f. Then

$$f(0) =$$

$f(0) = -f(-0) = -f(0)$

since $0 = -0$

(a) 1

(b) -1

(c) 0

(d) More information would be needed to determine $f(0)$.

x-axis Symmetry

Consider the relation $\sqrt[3]{(x-1)^2} + \sqrt[3]{y^2} = 1$. Note that if we replace y with $-y$ on the left side, we get

$$\sqrt[3]{(x-1)^2} + \sqrt[3]{(-y)^2} = \sqrt[3]{(x-1)^2} + \sqrt[3]{y^2}$$

So if (x, y) is on the graph of the relation, so is $(x, -y)$. Such a relation is said to have **symmetry with respect to the x-axis**—or just x-axis symmetry for short.

x-axis Symmetry

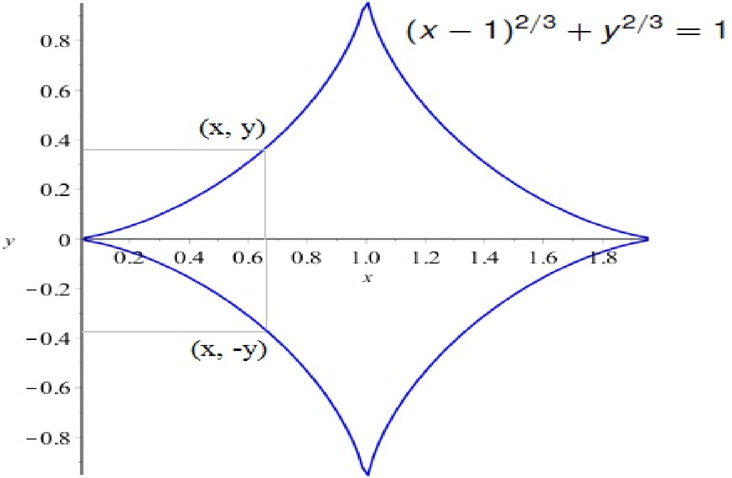

Figure: The part of the graph below the x-axis is the mirror image of the part above the x-axis.

Symmetry Checks

For a function or a relation given in terms of algebraic expressions, we can check for symmetry:

- **Even:** if replacing (x, y) with $(-x, y)$ results in the same formula (i.e. $f(-x) = f(x)$)
- **Odd:** if replacing (x, y) with $(-x, -y)$ results in the same foruma (i.e. $f(-x) = -f(x)$)
- **x-axis:** if replacing (x, y) with $(x, -y)$ results in the same formula.

Question

Jack and Diane are working together to graph a function. Jack thinks they should test the **function** for x-axis symmetry. Diane says it's not necessary to check the **function** for x-axis symmetry. Who is correct, and why?

(a) Jack is correct because all functions have x-axis symmetry.

(b) Jack is correct because some but not all functions have x-axis symmetry.

(c) Diane is correct because very few functions have x-axis symmetry.

(d) Diane is correct because no function can have x-axis symmetry.

violates the vertical line test!

Minute Exercise

Take a moment to write a sentence or two to explain why

- it IS possible for a function to have y-axis symmetry, but
- it's NOT possible for a function to have x-axis symmetry.