# August 29 MATH 1113 sec. 52 Fall 2018

## Section 2.5: Basic Transformations

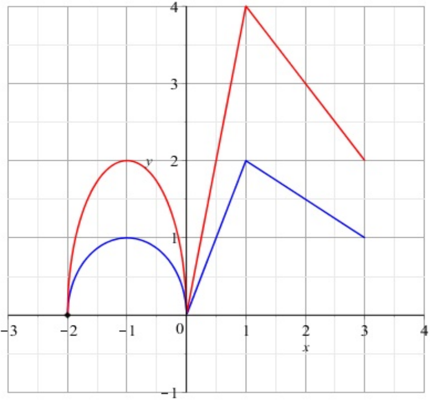
**Stretching and Shrinking** Since we already know that introducing a minus sign as in $f(-x)$ and $-f(x)$ results in a reflection, let's consider a positive number $a$ and investigate the relationship between the graph of $y = f(x)$ and each of

$$y = af(x), \quad \text{and} \quad y = f(ax).$$

The outcome depends on whether $a > 1$ or $0 < a < 1$.

Why aren't we bothering with the case $a = 1$?

# Vertical Stretch or Shrink: $y = af(x)$

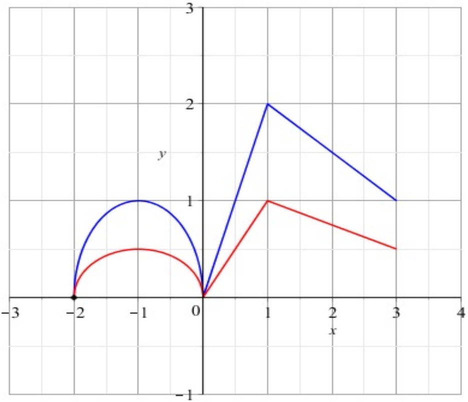



| $x$ | $f(x)$ |
|---|---|
| $-2$ | $0$ |
| $-1$ | $1$ |
| $0$ | $0$ |
| $1$ | $2$ |
| $2$ | $\frac{3}{2}$ |
| $3$ | $1$ |

| $x$ | $2f(x)$ |
|---|---|
| $-2$ | $2 \cdot 0 = 0$ |
| $-1$ | $2 \cdot 1 = 2$ |
| $0$ | $0$ |
| $1$ | $4$ |
| $2$ | $3$ |
| $3$ | $2$ |

Figure: $y = f(x)$ is in blue, and $y = 2f(x)$ is in red. Since $a = 2 > 1$, the graph is stretched vertically.

# Vertical Stretch or Shrink: $y = af(x)$




| $x$ | $f(x)$ |
|---|---|
| $-2$ | $0$ |
| $-1$ | $1$ |
| $0$ | $0$ |
| $1$ | $2$ |
| $2$ | $\frac{3}{2}$ |
| $3$ | $1$ |

| $x$ | $\frac{1}{2}f(x)$ |
|---|---|
| $-2$ | $\frac{1}{2}\cdot 0 = 0$ |
| $-1$ | $\frac{1}{2}\cdot 1 = \frac{1}{2}$ |
| $0$ | $0$ |
| $1$ | $1$ |
| $2$ | $\frac{3}{4}$ |
| $3$ | $\frac{1}{2}$ |

Figure: $y = f(x)$ is in blue, and $y = \frac{1}{2}f(x)$ is in red. Since $a = \frac{1}{2} < 1$, the graph is compressed vertically.

# Vertical Stretch or Shrink: $y = af(x)$

The graph of $y = af(x)$ is obtained from the graph of $y = f(x)$. If $a > 0$, then

$y = af(x)$ is stretched vertically if $a > 1$, and
$y = af(x)$ is shrunk (a.k.a. compressed) vertically if $0 < a < 1$.

If $a < 0$, then the stretch ($|a| > 1$) or shrink ($0 < |a| < 1$) is combined with a reflection in the $x$-axis.

# Horizontal Stretch or Shrink: $y = f(cx)$

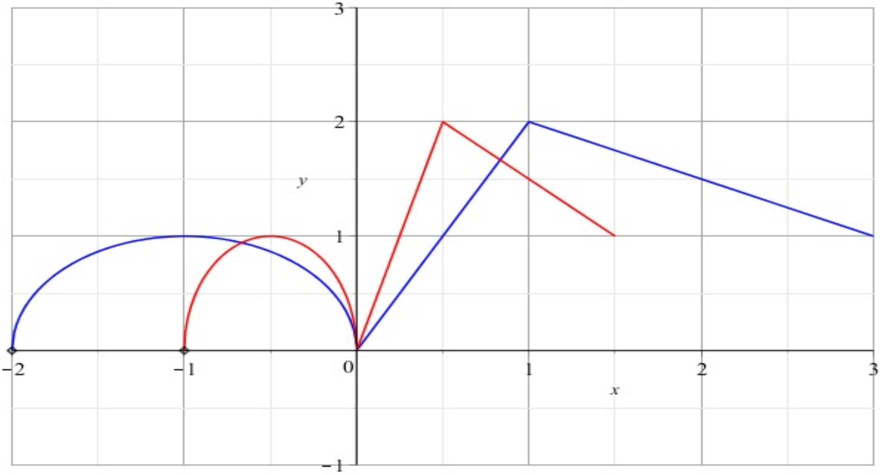



Figure: $y = f(x)$ is in blue, and $y = f(2x)$ is in red. Since $c = 2 > 1$, the graph is compressed horizontally.

# Horizontal Stretch or Shrink: $y = f(cx)$

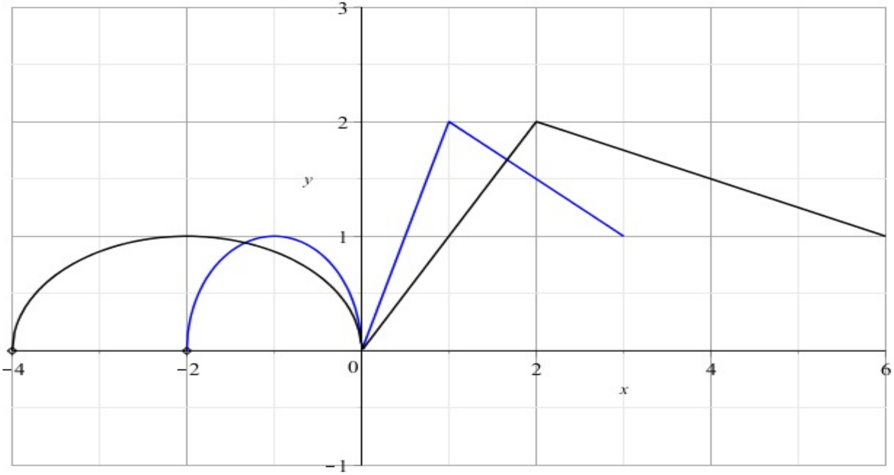



Figure: $y = f(x)$ is in blue, and $y = f\left(\frac{1}{2}x\right)$ is in black. Since $c = \frac{1}{2} < 1$, the graph is stretched horizontally.

# Horizontal Stretch or Shrink: $y = f(cx)$

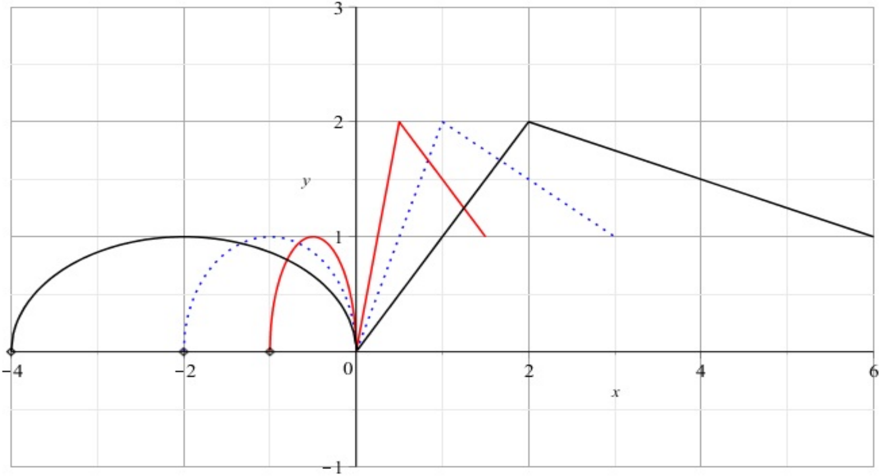



Figure: $y = f(x)$ is in blue dots. The compressed red curve is $y = f(2x)$, and the stretched black curve is $y = f\left(\frac{1}{2}x\right)$.

# Horizontal Stretch or Shrink: $y = f(cx)$

The examples given generalize except that we did not consider an example with $c < 0$. This combines the stretch/shrink with a reflection. We have the following result:

The graph of $y = f(cx)$ is obtained from the graph of $y = f(x)$. If $c > 0$, then

$y = f(cx)$ is shrunk (a.k.a. compressed) horizontally if $c > 1$, and
$y = f(cx)$ is stretched horizontally if $0 < c < 1$.

If $c < 0$, then the shrink ($|c| > 1$) or stretch ($0 < |c| < 1$) is combined with a reflection in the $y$-axis.

# Section 2.4: Symmetry

Consider the function $f(x) = 2x^2 + 1$. Suppose we wished to plot the new function $h(x) = f(-x)$. ← reflection in the y-axis

Find the formula for $h(x) = f(-x)$.

$$h(x) = f(-x) = 2(-x)^2 + 1 = 2(-x)(-x) + 1$$

$$= 2x^2 + 1 = f(x)$$

# Even Functions

For $f(x) = 2x^2 + 1$, we found that

$$f(-x) = 2(-x)^2 + 1 = 2x^2 + 1 = f(x).$$

Since the graph of $f(-x)$ is obtained from $f$ by reflection in the $y$-axis, and $f(-x)$ and $f(x)$ are the same for each $x$, it must be that

**the graph of $f$ is its own reflection in the $y$-axis!**

**Definition:** A function $f$ is called an **even function** if

$$f(-x) = f(x)$$

for each $x$ in its domain. We can say that such a function has **even symmetry**.

# Even Symmetry

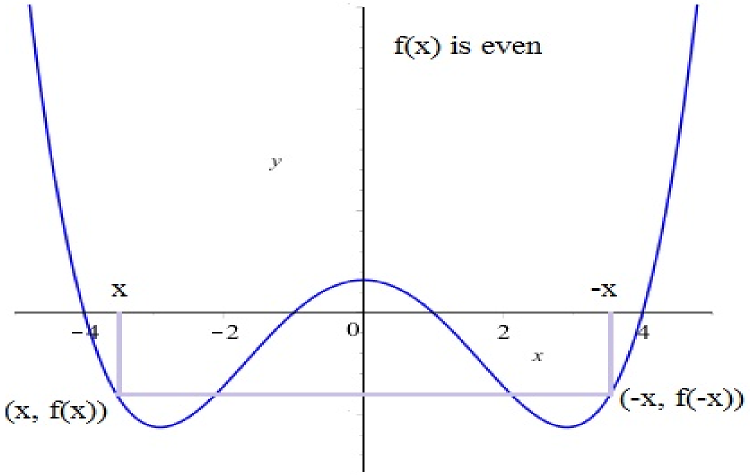



Figure: The graph to the left of the $y$-axis is the mirror image of the graph on the right side if a function is even.

# Question

Suppose $f$ is an even function whose graph passes through the point $(-2, 5)$. Which of the following points must be on the graph of $f$?

(a) $(5, -2)$

$f(-2) = 5$

(b) $(-2, -5)$

since $f$ is even

(c) $(2, -5)$

$f(-(-2)) = 5$

$f(2) = 5$

(d) $(2, 5)$

(e) There isn't enough information to determine if any of the above **must** be on the graph.

# Symmetry

Consider the function $f(x) = x - \frac{1}{2}x^3$, and let $g(x) = f(-x)$.

Find the formula for $g(x)$.

$$g(x) = f(-x) = (-x) - \frac{1}{2}(-x)^3 = -x - \frac{1}{2}(-x)(-x)(-x)$$

$$= -x - \frac{1}{2}(-x^3) = -x + \frac{1}{2}x^3$$

$$= -(x - \frac{1}{2}x^3) = -f(x)$$

↑ reflection in y axis

↑ reflection in x-axis

# Odd Functions

For $f(x) = x - \frac{1}{2}x^3$, we found that

$$f(-x) = (-x) - \frac{1}{2}(-x)^2 = -x + \frac{1}{2}x^3 = -\left(x - \frac{1}{2}x^3\right) = -f(x).$$

So $f(-x)$ is the reflection in the $y$-axis, and it's equal to the reflection in the $x$-axis. That is

**the reflection of $f$ in the $y$-axis is its reflection in the $x$-axis!**

**Definition:** A function $f$ is called an **odd function** if

$$f(-x) = -f(x)$$

for each $x$ in its domain. We can say that such a function has **odd symmetry**.

# Odd Symmetry

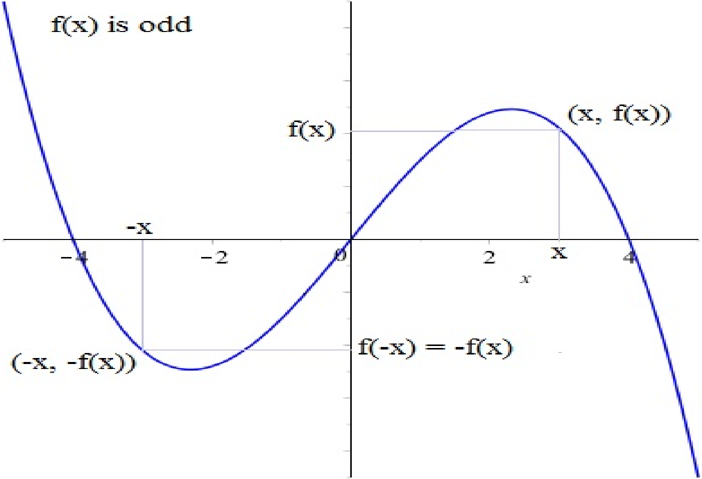



Figure: The graph of $f$ to the left of the $y$-axis can be obtained by reflecting the graph on the right twice—through the $y$-axis and then the $x$-axis.

# Even and Odd Symmetry

- Polynomials with only even powers (including 0) are even functions. Polynomials with only odd powers are odd functions.
- Even symmetry is called symmetry with respect to the $y$-axis.
- Odd symmetry is called symmetry with respect to the origin.
- Not all functions have symmetry (for example polynomials with both even and odd power terms). Some important functions have known symmetry.

# Question

Suppose $f$ is an **odd** function and that 0 is in the domain of $f$. Then

$$f(0) = -f(-0) = -f(0)$$

(a) 1

(b) $-1$

(c) 0

(d) More information would be needed to determine $f(0)$.

$f(x) = -f(-x)$

for $f$ odd

# $x$-axis Symmetry

Consider the relation $\sqrt[3]{(x-1)^2}+\sqrt[3]{y^2}=1$. Note that if we replace $y$ with $-y$ on the left side, we get

$$\sqrt[3]{(x-1)^2}+\sqrt[3]{(-y)^2}=\sqrt[3]{(x-1)^2}+\sqrt[3]{y^2}$$

So if $(x,y)$ is on the graph of the relation, so is $(x,-y)$. Such a relation is said to have **symmetry with respect to the $x$-axis**—or just $x$-axis symmetry for short.

# $x$-axis Symmetry

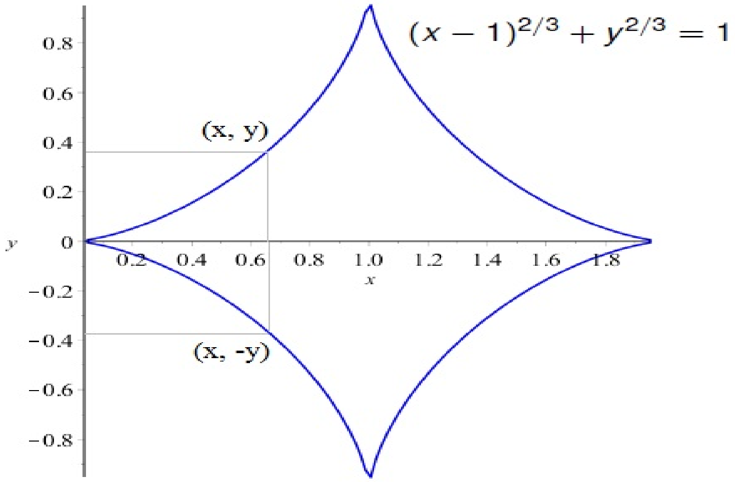



Figure: The part of the graph below the $x$-axis is the mirror image of the part above the $x$-axis.

# Symmetry Checks

For a function or a relation given in terms of algebraic expressions, we can check for symmetry:

- **Even:** if replacing $(x, y)$ with $(-x, y)$ results in the same formula (i.e. $f(-x) = f(x)$)
- **Odd:** if replacing $(x, y)$ with $(-x, -y)$ results in the same foruma (i.e. $f(-x) = -f(x)$)
- **$x$-axis:** if replacing $(x, y)$ with $(x, -y)$ results in the same formula.

# Question

Jack and Diane are working together to graph a function. Jack thinks they should test the **function** for $x$-axis symmetry. Diane says it's not necessary to check the **function** for $x$-axis symmetry. Who is correct, and why?

(a) Jack is correct because all functions have $x$-axis symmetry.

(b) Jack is correct because some but not all functions have $x$-axis symmetry.

(c) Diane is correct because very few functions have $x$-axis symmetry.

(d) Diane is correct because no function can have $x$-axis symmetry.

violates vertical line test