August 29 Math 1190 sec. 51 Fall 2016

Section 1.3: Continuity

We have seen that their may or may not be a relationship between the quantities

$$\lim_{x \to c} f(x) \quad \text{and} \quad f(c).$$

One or the other (or both) may fail to exist. And even if both exist, they need not be equivalent.

We've also seen that for polynomials at least, that the limit at a point is the same as the function value at that point. Here, we explore this property that polynomials (and lots of other functions, but not all) share.

Definition: Continuity at a Point

Definition: A function f is continuous at a number c if

$$\lim_{x \to c} f(x) = f(c).$$

Note that three properties are contained in this statement:
(1) $f(c)$ is defined (i.e. c is in the domain of f),

(2) $\lim_{x \to c} f(x)$ exists, and

(3) the limit actually equals the function value.

If a function f is not continuous at c, we may say that f is **discontinuous** at c

Polynomials and Rational Functions

In the previous section, we saw that:

If P is any polynomial and c is any real number, then $\lim_{x\to c} P(x) = P(c)$, and

If R is any rational function and c is any number in the domain of R, then $\lim_{x\to c} R(x) = R(c)$.

Conclusion **Theorem:** Every rational function* is continuous at each number in its domain.

*Note that polynomials can be lumped in to the set of all rational functions.

Examples: Determine where each function is discontinuous.

(a) $f(x) = \dfrac{x^2 - 4x}{x^2 - 3x - 4}$

f is a rational function. So it's continuous on its domain.

So its only discontinuous at points not in its domain — i.e. where the denominator is zero.

$x^2 - 3x - 4 = 0 \Rightarrow (x-4)(x+1) = 0 \Rightarrow x = 4$ or $x = -1$

f is discontinuous @ 4 and -1.

(b) $f(x) = \begin{cases} 2x, & x < 1 \\ x^2 + 1, & 1 \leq x < 2 \\ 3, & x \geq 2 \end{cases}$

Each piece is polynomial
It's continuous except
possibly where the
pieces change.

Consider 1:

$f(1) = 1^2 + 1 = 2$, 1 is in the domain and $f(1) = 2$.

$$\left.\begin{aligned} \lim_{x \to 1^+} f(x) &= \lim_{x \to 1^+} (x^2+1) = 1^2+1 = 2 \\ \lim_{x \to 1^-} f(x) &= \lim_{x \to 1^-} 2x = 2(1) = 2 \end{aligned}\right\} \lim_{x \to 1} f(x) = 2$$

Since $\lim_{x \to 1} f(x) = f(1)$ f is continuous @ 1.

Consider 2:

$$f(2) = 3$$

$$\lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} 3 = 3$$

$$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} (x^2+1) = 2^2+1 = 5$$

$$\lim_{x \to 2} f(x) \text{ DNE}$$

f is discontinuous @ 2.

Question

Determine whether f is continuous at 1 where $f(x) = \begin{cases} \frac{x^2-1}{x-1}, & x \neq 1 \\ 2, & x = 1 \end{cases}$

$f(1) = 2$

(a) No because $f(1)$ is not defined.

(b) Yes because all three conditions hold.

(c) No because $\lim_{x \to 1} f(x)$ doesn't exist.

(d) No because f is piecewise defined.

$\lim_{x \to 1} f(x) = \lim_{x \to 1} \frac{x^2-1}{x-1}$

$= \lim_{x \to 1} \frac{(x-1)(x+1)}{x-1}$

$= \lim_{x \to 1} x+1 = 1+1 = 2$

Removable and Jump Discontinuities

Definition: Let f be defined on an open interval containing c except possibly at c. If $\lim_{x \to c} f(x)$ exists, but f is discontinuous at c, then f has a **removable discontinuity** at c.

Definition: If $\lim_{x \to c^-} f(x) = L_1$ and $\lim_{x \to c^+} f(x) = L_2$ where $L_1 \neq L_2$ (i.e. both one sided limits exist but are different), then f has a **jump discontinuity** at c.

Removable and Jump Discontinuities

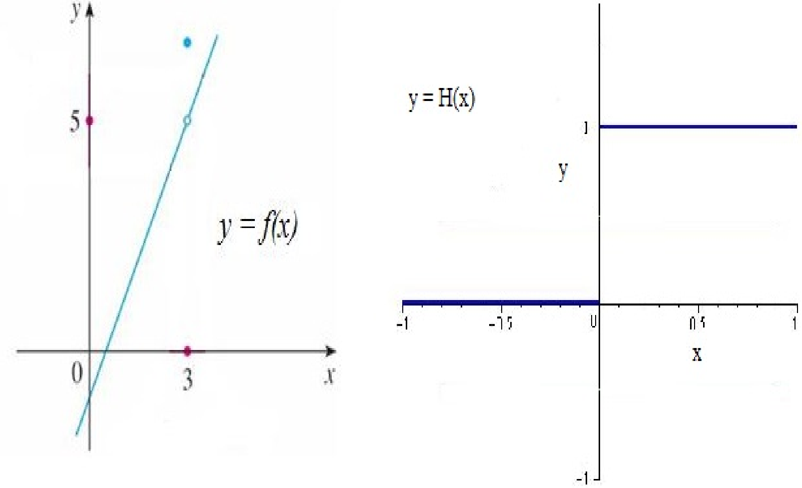


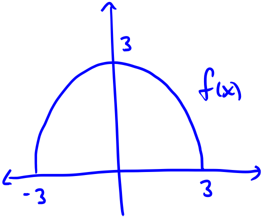
Figure: Example of a removable (left) discontinuity and a jump (right) discontinuity.

One Sided Continuity Example:

Consider the function $f(x) = \sqrt{9 - x^2}$. Plot a rough sketch of the graph of f, and determine its domain.

If $y = \sqrt{9-x^2} \Rightarrow y^2 = 9 - x^2 \Rightarrow x^2 + y^2 = 9$

top half of a circle radius 3 centered @ $(0,0)$



Domain: we need

$9 - x^2 \geq 0 \Rightarrow x^2 \leq 9$

$\Rightarrow \quad |x| \leq 3$

i.e. $\quad -3 \leq x \leq 3$

The domain is $[-3, 3]$.

$$f(x) = \sqrt{9 - x^2}$$

Note that f is continuous on $-3 < x < 3$. What can be said about

$$\lim_{x \to -3} f(x) \qquad \text{or} \qquad \lim_{x \to 3} f(x)?$$

Neither of these exist. f is not defined on an open interval containing -3 or 3.

The domain of f doesn't contain any numbers to the left of -3 or to the right of $+3$.

Continuity From the Left & Right

Definition: Let a function f be defined on an interval $[c, b)$. Then f is continuous from the right at c if

$$\lim_{x \to c^+} f(x) = f(c).$$

Let f be defined on an interval $(a, c]$. Then f is continuous from the left at c if

$$\lim_{x \to c^-} f(x) = f(c).$$

Example: $f(x) = \sqrt{9 - x^2}$

Since

$$\lim_{x \to -3^+} f(x) = f(-3) = 0 \qquad \lim_{x \to 3^-} f(x) = f(3) = 0$$

f is continuous from the right at -3 and continuous from the left at 3.
We can say that f is continuous at every value in $[-3, 3]$.

A Theorem on Continuous Functions

Theorem If f and g are continuous at c and for any constant k, the following are also continuous at c:

$$(i)\, f+g, \quad (ii)\, f-g, \quad (iii)\, kf, \quad (iv)\, fg, \quad \text{and} \quad (v)\, \frac{f}{g}, \text{ if } g(c) \neq 0.$$

In other words, if we combine continuous functions using addition, subtraction, multiplication, division, and using constant factors, the result is also continuous—provided of course that we don't introduce division by zero.

Questions

(1) **True or False** If f is continuous at 3 and g is continuous at 3, then it must be that

$$\lim_{x\to 3} f(x)g(x) = f(3)g(3).$$

True, products of cont. functions are cont.

(2) **True or False** If $f(2) = 1$ and $g(2) = 7$, then it must be that

$$\lim_{x\to 2} \frac{f(x)}{g(x)} = \frac{1}{7}.$$

false, f or g may be discontinuos @ 2.

Continuity is key. If f or g isn't continuous, the limit statement may fail.

Ex. $f(x) = \begin{cases} \dfrac{x^2-4}{x-2}, & x \neq 2 \\ 1, & x = 2 \end{cases}$ so $f(2) = 1$

$g(x) = 7$ so $g(2) = 7$

But $\lim_{x \to 2} f(x) = \lim_{x \to 2} \dfrac{x^2-4}{x-2} = \lim_{x \to 2} x+2 = 4$

f is not cont @ 2.

$$\lim_{x \to 2} \frac{f(x)}{g(x)} = \frac{\lim_{x \to 2} f(x)}{\lim_{x \to 2} g(x)} = \frac{4}{7} \text{ not } \frac{1}{7}!$$

Continuity on an Interval

Definition A function is continuous on an interval (a, b) if it is continuous at each point in (a, b). A function is continuous on an interval such as $(a, b]$ or $[a, b)$ or $[a, b]$ provided it is continuous on (a, b) and has one sided continuity at each included end point.

Graphically speaking, if $f(x)$ is continuous on an interval (a, b), then the curve $y = f(x)$ will have no holes or gaps.

Find all values of A such that f is continuous on $(-\infty, \infty)$.

$$f(x) = \begin{cases} x + A, & x < 2 \\ Ax^2 - 3, & 2 \le x \end{cases}$$

For any A, the pieces $y = x + A$ and $y = Ax^2 - 3$ are continuous (polynomials).

The only point @ issue is 2. We need $\lim_{x \to 2} f(x) = f(2)$.

$$f(2) = A(2)^2 - 3 = 4A - 3$$

$$\lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} (Ax^2 - 3) = 4A - 3$$

$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} (x+A) = 2+A$$

For the limit to exist and equal $f(2)$, the two one sided limits must be equal.

$$4A - 3 = 2 + A$$

right limit

left limit

$$3A = 5 \Rightarrow A = \frac{5}{3}$$

f is continuous @ 2 — and therefore on $(-\infty,\infty)$

if $A = \frac{5}{3}$.

$$f(x) = \begin{cases} x + \frac{5}{3}, & x < 2 \\ \frac{5}{3}x^2 - 3, & x \geq 2 \end{cases}$$