August 29 Math 1190 sec. 52 Fall 2016

Section 1.3: Continuity

Definition: A function f is continuous at a number c if

$$\lim_{x \to c} f(x) = f(c).$$

Note that three properties are contained in this statement:
(1) $f(c)$ is defined (i.e. c is in the domain of f),

(2) $\lim_{x \to c} f(x)$ exists, and

(3) the limit actually equals the function value.

If a function f is not continuous at c, we may say that f is **discontinuous** at c

Polynomials and Rational Functions

In the previous section, we saw that:

If P is any polynomial and c is any real number, then $\lim_{x \to c} P(x) = P(c)$, and

If R is any rational function and c is any number in the domain of R, then $\lim_{x \to c} R(x) = R(c)$.

Conclusion **Theorem:** Every rational function* is continuous at each number in its domain.

*Note that polynomials can be lumped in to the set of all rational functions.

Examples: Determine where each function is discontinuous.

(a) $f(x) = \dfrac{x^2 - 4x}{x^2 - 3x - 4}$

f is a rational function, hence it's continuous on its domain.

It's discontinuous at numbers not in its domain, and that's where its denominator is zero.

$x^2 - 3x - 4 = 0 \Rightarrow (x-4)(x+1) = 0 \Rightarrow x = 4$ or $x = -1$

so f is discontinuous @ 4 and -1.

(b) $f(x) = \begin{cases} 2x, & x < 1 \\ x^2 + 1, & 1 \le x < 2 \\ 3, & x \ge 2 \end{cases}$

The pieces are polynomial and are therefore continuous at every x. The possible points for discontinuity are where f changes @ 1 and 2.

Consider 1:

$f(1) = 1^2 + 1 = 2$

$$\left.\begin{aligned} \lim_{x \to 1^+} f(x) &= \lim_{x \to 1^+} (x^2+1) = 1^2+1 = 2 \\ \lim_{x \to 1^-} f(x) &= \lim_{x \to 1^-} 2x = 2(1) = 2 \end{aligned}\right\} \lim_{x \to 1} f(x) = 2$$

$\lim_{x \to 1} f(x) = f(1)$ f is continuous @ 1.

Consider 2:

$f(2) = 3$

$\lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} 3 = 3$

$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} (x^2+1) = 2^2+1 = 5$

$\Bigg\}$ one sided limits disagree

$\lim_{x \to 2} f(x)$ DNE

f is discontinuous @ 2.

Question

Determine whether f is continuous at 1 where $f(x) = \begin{cases} \frac{x^2-1}{x-1}, & x \neq 1 \\ 2, & x = 1 \end{cases}$

$f(1) = 2$

(a) No because $f(1)$ is not defined.

(b) Yes because all three conditions hold.

(c) No because $\lim_{x\to 1} f(x)$ doesn't exist.

(d) No because f is piecewise defined.

$$\lim_{x\to 1} f(x) = \lim_{x\to 1} \frac{x^2-1}{x-1}$$

$$= \lim_{x\to 1} \frac{(x-1)(x+1)}{x-1}$$

$$= \lim_{x\to 1} x+1 = 1+1 = 2$$

note $\lim_{x\to 1} f(x) = f(1)$

Removable and Jump Discontinuities

Definition: Let f be defined on an open interval containing c except possibly at c. If $\lim_{x \to c} f(x)$ exists, but f is discontinuous at c, then f has a **removable discontinuity** at c.

Definition: If $\lim_{x \to c^-} f(x) = L_1$ and $\lim_{x \to c^+} f(x) = L_2$ where $L_1 \neq L_2$ (i.e. both one sided limits exist but are different), then f has a **jump discontinuity** at c.

Removable and Jump Discontinuities

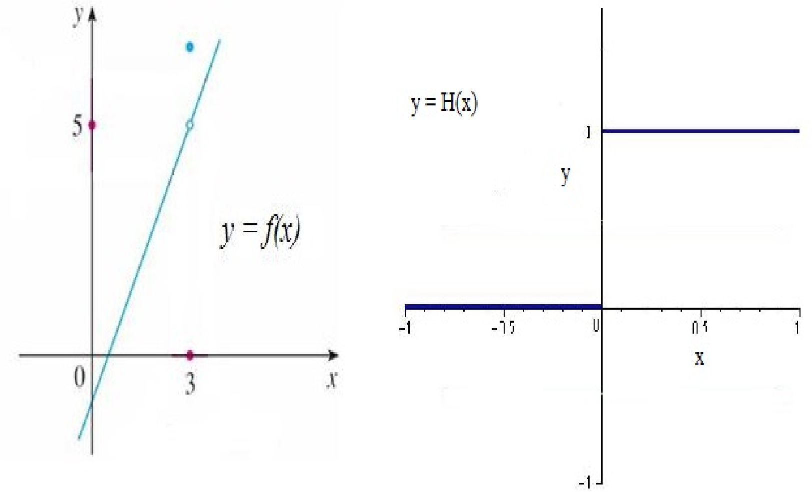


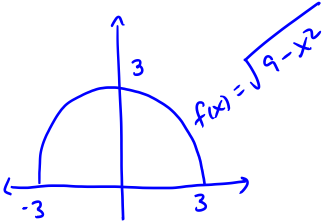
Figure: Example of a removable (left) discontinuity and a jump (right) discontinuity.

One Sided Continuity Example:

Consider the function $f(x) = \sqrt{9 - x^2}$. Plot a rough sketch of the graph of f, and determine its domain.

$y = \sqrt{9 - x^2} \Rightarrow y^2 = 9 - x^2 \Rightarrow x^2 + y^2 = 9$

Top half of a circle of radius 3 centered @ (0,0).



Domain: We require

$9 - x^2 \geq 0 \Rightarrow x^2 \leq 9$

$\Rightarrow \sqrt{x^2} \leq \sqrt{9} \Rightarrow |x| \leq 3$

$\Rightarrow -3 \leq x \leq 3$

The domain of f is the interval $[-3, 3]$.

$$f(x) = \sqrt{9 - x^2}$$

Note that f is continuous on $-3 < x < 3$. What can be said about

$$\lim_{x \to -3} f(x) \qquad \text{or} \qquad \lim_{x \to 3} f(x)?$$

Neither of these limits exists.

f is not defined on any open interval containing either -3 or 3.

f is not defined to the left of -3 nor to the right of 3.

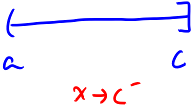
Continuity From the Left & Right

Definition: Let a function f be defined on an interval $[c, b)$. Then f is continuous from the right at c if

$$\lim_{x \to c^+} f(x) = f(c).$$

Let f be defined on an interval $(a, c]$. Then f is continuous from the left at c if

$$\lim_{x \to c^-} f(x) = f(c).$$



Example: $f(x) = \sqrt{9 - x^2}$

Since

$$\lim_{x \to -3^+} f(x) = f(-3) = 0 \qquad \lim_{x \to 3^-} f(x) = f(3) = 0$$

f is continuous from the right at -3 and continuous from the left at 3.
We can say that f is continuous at every value in $[-3, 3]$.

A Theorem on Continuous Functions

Theorem If f and g are continuous at c and for any constant k, the following are also continuous at c:

$$(i)\, f+g, \quad (ii)\, f-g, \quad (iii)\, kf, \quad (iv)\, fg, \quad \text{and} \quad (v)\, \frac{f}{g}, \text{ if } g(c) \neq 0.$$

In other words, if we combine continuous functions using addition, subtraction, multiplication, division, and using constant factors, the result is also continuous—provided of course that we don't introduce division by zero.

Questions

(1) **True or False** If f is continuous at 3 and g is continuous at 3, then it must be that

$$\lim_{x \to 3} f(x)g(x) = f(3)g(3).$$

True, the product of cont. fnct. is continuous,

(2) **True or False** If $f(2) = 1$ and $g(2) = 7$, then it must be that

$$\lim_{x \to 2} \frac{f(x)}{g(x)} = \frac{1}{7}.$$

False If f or g is discontinuous @ 2, the limit might not be $f(2)/g(2)$.

Continuity on an Interval

Definition A function is continuous on an interval (a, b) if it is continuous at each point in (a, b). A function is continuous on an interval such as $(a, b]$ or $[a, b)$ or $[a, b]$ provided it is continuous on (a, b) and has one sided continuity at each included end point.

Graphically speaking, if $f(x)$ is continuous on an interval (a, b), then the curve $y = f(x)$ will have no holes or gaps.

Find all values of A such that f is continuous on $(-\infty, \infty)$.

$$f(x) = \begin{cases} x + A, & x < 2 \\ Ax^2 - 3, & 2 \leq x \end{cases}$$

Both pieces are polynomials and are hence continuous at all x. So the only point at which continuity could fail is @ $x = 2$.

For continuity @ 2, we require $\lim_{x \to 2} f(x) = f(2)$.

$f(2) = A(2)^2 - 3 = 4A - 3$

$$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} (Ax^2-3) = 4A-3$$

$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} (x+A) = 2+A$$

$\lim_{x\to 2^+} f(x) = f(2)$, we also require

$\lim_{x\to 2^-} f(x) = f(2)$

This gives the equation

$$2+A = 4A-3$$

left limit

right limit and $f(2)$

$$5 = 3A \Rightarrow A = \frac{5}{3}$$

f is continuous on $(-\infty, \infty)$ if $A = 5/3$.

f would be

$$f(x) = \begin{cases} x + \frac{5}{3}, & x < 2 \\ \frac{5}{3}x^2 - 3, & x \geq 2 \end{cases}$$