August 29 Math 2306 sec. 53 Fall 2018

Section 4: First Order Equations: Bernoulli Equations

Suppose $P(x)$ and $f(x)$ are continuous on some interval (a, b) and n is a real number different from 0 or 1 (not necessarily an integer). An equation of the form

$$\frac{dy}{dx} + P(x)y = f(x)y^n$$

is called a **Bernoulli** equation.

Observation: This equation has the flavor of a linear ODE, but since $n \neq 0, 1$ it is necessarily nonlinear. So our previous approach involving an integrating factor does not apply directly. Fortunately, we can use a change of variables to obtain a related linear equation.

Solving the Bernoulli Equation

$$\frac{dy}{dx} + P(x)y = f(x)y^n$$

We'll introduce a new dependent variable u.

Set $u = y^{1-n}$. Note that

$$\frac{du}{dx} = (1-n)y^{1-n-1}\frac{dy}{dx} = (1-n)y^{-n}\frac{dy}{dx}$$

Solving for $\frac{dy}{dx} = \frac{1}{1-n}y^n\frac{du}{dx}$

Sub this into the ODE

$$\frac{1}{1-n}y^n\frac{du}{dx} + P(x)y = f(x)y^n$$

Multiply by $\frac{1-n}{y^n}$

$$\frac{du}{dx} + \frac{1-n}{y^n} P(x)\, y = \frac{1-n}{\cancel{y^n}} f(x) \cancel{y^n}$$

$$\frac{du}{dx} + (1-n) P(x)\, y^{1-n} = (1-n) f(x)$$

Recall $u = y^{1-n}$

The above is

$$\frac{du}{dx} + (1-n) P(x)\, u = (1-n) f(x)$$

This is the linear ODE for u

$$\frac{du}{dx} + P_1(x)\,u = f_1(x)$$

where $P_1(x) = (1-n)\,P(x)$ and

$$f_1(x) = (1-n)\,f(x)$$

Solve using an integrating factor to get u, then from

$u = y^{1-n}$, we get $y = u^{\frac{1}{1-n}}$.

Example

Solve the initial value problem $y' - y = -e^{2x}y^3$, subject to $y(0) = 1$.

Here $n = 3$. So $u = y^{1-3} = y^{-2}$

$$\frac{du}{dx} = -2y^{-3}\frac{dy}{dx} \implies \frac{dy}{dx} = \frac{-1}{2}y^3\frac{du}{dx}$$

Substitute

$$\frac{-1}{2}y^3\frac{du}{dx} - y = -e^{2x}y^3$$

Divide out $\frac{-1}{2}y^3$

$$\frac{du}{dx} - \left(\frac{-2}{y^3}\right)y = \left(\frac{-2}{y^3}\right)\left(-e^{2x}\right)y^3$$

$$\frac{du}{dx} + 2y^{-2} = 2e^{2x}$$

Since $y^{-2} = u$

$$\frac{du}{dx} + 2u = 2e^{2x}$$

For u, the $P(x)=2$. The integrating factor is

$$\mu = e^{\int P(x)\,dx} = e^{\int 2\,dx} = e^{2x}$$

$$e^{2x}\left(\frac{du}{dx} + 2u\right) = e^{2x}\left(2e^{2x}\right)$$

$$\frac{d}{dx}\left(e^{2x}u\right) = 2e^{4x}$$

$$\int \frac{d}{dx}\left(e^{2x}u\right)dx = \int 2e^{4x}\,dx$$

$$e^{2x}u = \frac{1}{2}e^{4x} + C$$

$$u = \frac{\frac{1}{2}e^{4x} + C}{e^{2x}} = \frac{1}{2}e^{2x} + Ce^{-2x}$$

From $u = y^{-2}$, $y = u^{-1/2}$

So $$y = \left(\frac{1}{2}e^{2x} + Ce^{-2x}\right)^{-1/2}.$$

Now apply the IC $y(0) = 1$

$$1 = \left(\frac{1}{2}e^{0} + Ce^{0}\right)^{-1/2} \Rightarrow 1 = \left(\frac{1}{2} + C\right)^{-1/2}$$

$$1^{-2} = \left(\left(\frac{1}{2} + C\right)^{-1/2}\right)^{-2} \Rightarrow 1 = \frac{1}{2} + C$$

$$C = \frac{1}{2}$$

The solution to the IVP is

$$\boxed{y = \left(\frac{1}{2}e^{2x} + \frac{1}{2}e^{-2x}\right)^{-1/2}}$$