Section 4: First Order Equations: Bernoulli Equations

Suppose $P(x)$ and $f(x)$ are continuous on some interval $(a, b)$ and $n$ is a real number different from 0 or 1 (not necessarily an integer). An equation of the form

$$\frac{dy}{dx} + P(x)y = f(x)y^n$$

is called a Bernoulli equation.

Observation: This equation has the flavor of a linear ODE, but since $n \neq 0, 1$ it is necessarily nonlinear. So our previous approach involving an integrating factor does not apply directly. Fortunately, we can use a change of variables to obtain a related linear equation.
Solving the Bernoulli Equation

\[
\frac{dy}{dx} + P(x)y = f(x)y^n
\]  \(1\)

Let the new variable \( u = y^{1-n} \). Then

\[
\frac{du}{dx} = (1-n) y^{-n-1} \frac{dy}{dx} = (1-n) y^{-n} \frac{dy}{dx}
\]

\[
\Rightarrow \quad \frac{du}{dx} = \frac{y^n}{1-n} \frac{du}{dx} \quad \left\langle \text{replace } \frac{dy}{dx} \text{ with this} \right\rangle
\]

\[
\frac{y^n}{1-n} \frac{du}{dx} + P(x) y^{1-n} = f(x) y^n
\]

Divide out \( \frac{y^n}{1-n} \)
\[
\frac{du}{dx} + (1-n) P(x) \frac{y}{y^n} = (1-n) f(x) \frac{y^n}{y^n}
\]

Note: \[\frac{y}{y^n} = y^{1-n} = u\]

\[
\frac{du}{dx} + (1-n) P(x) u = (1-n) f(x)
\]

Looks like \[\frac{du}{dx} + P_1(x) u = f_1(x)\]

where \[P_1 = (1-n) P\] and \[f_1 = (1-n) f\]
Solve this 1st order linear equation for \( u \). Then since

\[
\begin{align*}
&u = \frac{y}{1 - w} \\
y &= u \\
&\frac{1}{1 - w}
\end{align*}
\]
Example

Solve the initial value problem $y' - y = -e^{2x}y^3$, subject to $y(0) = 1$. 

$n = 3$, so $u = y^3 = y^2$.

$$\frac{du}{dx} = -2y \frac{dy}{dx} \Rightarrow \frac{1}{2} y^3 \frac{du}{dx} = \frac{dy}{dx}$$

Substitute

$$-\frac{1}{2} y^3 \frac{du}{dx} - y = -e^{2x} y^3$$

Divide \( \frac{1}{2} y^3 \)

$$\frac{du}{dx} - (-2) \frac{y}{y^3} = (-2)(-e^{2x}) \frac{y^3}{y^3}$$

$$\frac{y}{y^3} = y^{-2} = u$$
Solve \( \frac{du}{dx} + 2u = 2e^{2x} \)

\( P(x) = 2 \implies \mu = e^{\int P(x)dx} = e^{\int 2dx} = e^2 \)

Multiply by \( \mu \)

\[ e^2 \frac{du}{dx} + 2e^2 u = 2e^2 \cdot e^{2x} \]

\[ e^x \frac{du}{dx} + 2e^x u = 2e^{2x} 
\]

\[ \frac{d}{dx} [e^x u] = 2e^{4x} \]

\[ \int \frac{d}{dx} [e^x u] \, dx = \int 2e^{4x} \, dx \]

\[ e^x u = 2 \left( \frac{1}{4} e^{4x} \right) + C \]
\[
e^{2x} u = \frac{1}{2} e^{4x} + C
\]

\[
u = \frac{1}{2} e^{4x} + C \frac{e^{2x}}{e^{2x}}
\]

\[
u = \frac{1}{2} e^{2x} + Ce^{-2x}
\]

\[
sin u \quad u = -\frac{1}{2} \quad y = u^2 = \frac{1}{16
\]

\[
y = \frac{1}{\sqrt{\frac{1}{2} e^{2x} + Ce^{-2x}}}
\]
Impose the condition $y(0) = 1$

$$y(0) = \frac{1}{\sqrt{\frac{1}{2} e^0 + C e^0}} = 1$$

$$\Rightarrow \sqrt{\frac{1}{2} e^0 + C e^0} = 1$$

$$\frac{1}{2} + C = 1^2 = 1 \Rightarrow C = \frac{1}{2}$$

Finally,

$$y = \frac{1}{\sqrt{\frac{1}{2} e^{2x} + \frac{1}{2} e^{-2x}}}$$
we can clear the fractions by multiplying by \( \frac{\sqrt{2}}{\sqrt{2}} \) to get:

\[
y = \frac{1}{\sqrt{\frac{1}{2} e^{2x} + \frac{1}{2} e^{-2x}}} \cdot \frac{\sqrt{2}}{\sqrt{2}}
\]

\[
\theta = \frac{\sqrt{2}}{\sqrt{e^{2x} + e^{-2x}}}
\]
Partial Derivatives

If $F(x, y)$ is a function of two variables, $x$ and $y$, $F$ may be differentiable with respect to one (or the other or both). A derivative with respect to one variable is called a *partial derivative*, and the partial symbol $\partial$ is used to denote this. The other variable is held constant. For example,

$$F(x, y) = e^{x^2} \cos(y)$$

has first partial derivatives

$$\frac{\partial F}{\partial x} = 2xe^{x^2} \cos(y) \quad \text{and} \quad \frac{\partial F}{\partial y} = -e^{x^2} \sin(y).$$

If they exist, such a function will have four second partial derivatives

$$\frac{\partial^2 F}{\partial y \partial x}, \quad \frac{\partial^2 F}{\partial x \partial y}, \quad \frac{\partial^2 F}{\partial x^2}, \quad \text{and} \quad \frac{\partial^2 F}{\partial y^2}.$$
Exact Equations

We considered first order equations of the form

\[ M(x, y) \, dx + N(x, y) \, dy = 0. \]  \hspace{1cm} (2)

The left side is called a *differential form*. We will assume here that \( M \) and \( N \) are continuous on some (shared) region in the plane.

**Definition:** The equation (2) is called an **exact equation** on some rectangle \( R \) if there exists a function \( F(x, y) \) such that

\[
\frac{\partial F}{\partial x} = M(x, y) \quad \text{and} \quad \frac{\partial F}{\partial y} = N(x, y)
\]

for every \((x, y)\) in \( R \).
**Exact Equation Solution**

If \( M(x, y) \, dx + N(x, y) \, dy = 0 \) happens to be exact, then it is equivalent to

\[
\frac{\partial F}{\partial x} \, dx + \frac{\partial F}{\partial y} \, dy = 0
\]

This implies that the function \( F \) is constant on \( R \) and solutions to the DE are given by the relation

\[
F(x, y) = C
\]
Recognizing Exactness

There is a theorem from calculus that ensures that if a function $F$ has first partials on a domain, and if those partials are continuous, then the second mixed partials are equal. That is,

$$\frac{\partial^2 F}{\partial y \partial x} = \frac{\partial^2 F}{\partial x \partial y}.$$ 

If it is true that

$$\frac{\partial F}{\partial x} = M \quad \text{and} \quad \frac{\partial F}{\partial y} = N$$

this provides a condition for exactness, namely

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$$
Exact Equations

**Theorem:** Let $M$ and $N$ be continuous on some rectangle $R$ in the plane. Then the equation

$$M(x, y) \, dx + N(x, y) \, dy = 0$$

is exact if and only if

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$$
Example
Show that the equation is exact and obtain a family of solutions.

\[(2xy - \sec^2 x) \, dx + (x^2 + 2y) \, dy = 0\]

\[M(x,y) = 2xy - \sec^2 x \quad \text{and} \quad N(x,y) = x^2 + 2y\]

\[\frac{\partial M}{\partial y} = 2x \cdot 1 - 0 = 2x \quad \frac{\partial N}{\partial x} = 2x + 0 = 2x\]

\[\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}\]

so the equation is exact.

Solutions are given by \(F(x,y) = C\) where
\[ \frac{\partial F}{\partial x} = M(x,y) = 2xy - \sec^2 x \quad \text{and} \]
\[ \frac{\partial F}{\partial y} = N(x,y) = x^2 + 2y \]

\[ F(x,y) = \int \frac{\partial F}{\partial x} \, dx = \int (2xy - \sec^2 x) \, dx \]

\[ = 2y \cdot \frac{x^2}{2} - \tan x + g(y) \]

\[ = x^2y - \tan x + g(y) \]

* The "constant" of integration can be a function of \( y \). *
we know

1. \( F(x, y) = x^2y - \tan x + g(y) \)

and 2. \( \frac{\partial F}{\partial y} = x^2 + 2y \)

From 0 \( \frac{\partial F}{\partial y} = x^2 \cdot 1 - 0 + g'(y) \)

\[ = x^2 + g'(y) \]

Matching, \( g'(y) = 2y \)

an antiderivative is \( y^2 \)
So \( F(x, y) = x^2 y - \tan x + y^2 \)

All solutions are given implicitly by

\[ x^2 y - \tan x + y^2 = C \]