August 24 Math 2306 sec. 57 Fall 2017

Section 4: First Order Equations: Bernoulli Equations

Suppose $P(x)$ and $f(x)$ are continuous on some interval (a, b) and n is a real number different from 0 or 1 (not necessarily an integer). An equation of the form

$$\frac{dy}{dx} + P(x)y = f(x)y^n$$

is called a **Bernoulli** equation.

Observation: This equation has the flavor of a linear ODE, but since $n \neq 0, 1$ it is necessarily nonlinear. So our previous approach involving an integrating factor does not apply directly. Fortunately, we can use a change of variables to obtain a related linear equation.

Solving the Bernoulli Equation

$$\frac{dy}{dx} + P(x)y = f(x)y^n \tag{1}$$

Let the new variable $u = y^{1-n}$. Find $\frac{dy}{dx}$:

$$\frac{du}{dx} = (1-n)\, y^{1-n-1} \frac{dy}{dx} = (1-n)\, y^{-n} \frac{dy}{dx}$$

$$\Rightarrow \quad \frac{y^n}{1-n} \frac{du}{dx} = \frac{dy}{dx}$$

substitute into the ODE

$$\frac{y^n}{1-n} \frac{du}{dx} + P(x)\, y = f(x)\, y^n$$

Divide by $\frac{y^n}{1-n}$

$$\frac{du}{dx} + (1-n)P(x)\,\frac{y}{y^n} = (1-n)f(x)\,\frac{y^n}{y^n}$$

Note $\frac{y}{y^n} = y^{1-n} = u$

The ODE in u is

$$\frac{du}{dx} + (1-n)P(x)\,u = (1-n)f(x)$$

This is linear in u

$$\frac{du}{dx} + P_1(x)\,u = f_1(x) \quad \text{where}$$

$$P_1 = (1-n)P \quad \text{and} \quad f_1 = (1-n)f$$

Solve this for u. Then since

$$u = y^{1-n}$$

$$y = u^{\frac{1}{1-n}}$$

Example

Solve the initial value problem $y' - y = -e^{2x}y^3$, subject to $y(0) = 1$.

Here, $n=3$, so $u = y^{1-3} = y^{-2}$.

$$\frac{du}{dx} = -2y^{-3}\frac{dy}{dx} \Rightarrow -\frac{1}{2}y^3\frac{du}{dx} = \frac{dy}{dx}$$

Substitute

$$-\frac{1}{2}y^3\frac{du}{dx} - y = -e^{2x}y^3$$

$$\frac{du}{dx} - (-2)\frac{y}{y^3} = (-2)\left(-e^{2x}\right)\frac{y^3}{y^3}$$

$$\frac{y}{y^3} = y^{-2} = u$$

$$\frac{du}{dx} + 2u = 2e^{2x}$$ solve this

$$P(x) = 2, \quad \mu = e^{\int P(x)\,dx} = e^{\int 2\,dx} = e^{2x}$$

$$e^{2x}\frac{du}{dx} + 2e^{2x}u = 2e^{2x}\cdot e^{2x}$$

$$\frac{d}{dx}\left[e^{2x}u\right] = 2e^{4x}$$

$$\int \frac{d}{dx}\left[e^{2x}u\right]dx = \int 2e^{4x}\,dx$$

$$e^{2x}u = 2\left(\frac{1}{4}e^{4x}\right) + C$$

$$* \int e^{ax}\,dx = \frac{1}{a}e^{ax} + C$$

for constant $a \neq 0$

$$u = \frac{\frac{1}{2}e^{4x} + C}{e^{2x}}$$

$$u = \frac{1}{2}e^{2x} + Ce^{-2x}$$

Find y: $u = y^{-2}$ so $y = u^{-\frac{1}{2}} = \frac{1}{\sqrt{u}}$

$$y = \frac{1}{\sqrt{\frac{1}{2}e^{2x} + Ce^{-2x}}}$$

Now impose the condition $y(0) = 1$

$$y(0) = \frac{1}{\sqrt{\frac{1}{2}e^0 + Ce^0}} = 1$$

Solve for C

$$\sqrt{\frac{1}{2} + C} = 1$$

$$\frac{1}{2} + C = 1^2 = 1 \implies C = \frac{1}{2}$$

The solution to the IVP is

$$y = \frac{1}{\sqrt{\frac{1}{2}e^{2x} + \frac{1}{2}e^{-2x}}}$$

To avoid complex fractions multiply by $\frac{\sqrt{2}}{\sqrt{2}}$

$$y = \frac{1}{\sqrt{\frac{1}{2}e^{2x} + \frac{1}{2}e^{-2x}}}\left(\frac{\sqrt{2}}{\sqrt{2}}\right)$$

$$\boxed{y = \frac{\sqrt{2}}{\sqrt{e^{2x} + e^{-2x}}}}$$

Partial Derivatives

If $F(x, y)$ is a function of two variables, x and y, F may be differentiable with respect to one (or the other or both). A derivative with respect to one variable is called a *partial derivative*, and the partial symbol ∂ is used to denote this. The other variable is held constant. For example,

$$F(x, y) = e^{x^2} \cos(y)$$

has first partial derivatives

$$\frac{\partial F}{\partial x} = 2xe^{x^2} \cos(y) \quad \text{and} \quad \frac{\partial F}{\partial y} = -e^{x^2} \sin(y).$$

If they exist, such a function will have four second partial derivatives

$$\frac{\partial^2 F}{\partial y \partial x}, \quad \frac{\partial^2 F}{\partial x \partial y}, \quad \frac{\partial^2 F}{\partial x^2}, \quad \text{and} \quad \frac{\partial^2 F}{\partial y^2}.$$

Exact Equations

We considered first order equations of the form

$$M(x,y)\,dx + N(x,y)\,dy = 0. \tag{2}$$

The left side is called a *differential form*. We will assume here that M and N are continuous on some (shared) region in the plane.

Definition: The equation (2) is called an **exact equation** on some rectangle R if there exists a function $F(x,y)$ such that

$$\frac{\partial F}{\partial x} = M(x,y) \quad \text{and} \quad \frac{\partial F}{\partial y} = N(x,y)$$

for every (x,y) in R.

Exact Equation Solution

If $M(x,y)\,dx + N(x,y)\,dy = 0$ happens to be exact, then it is equivalent to

$$\frac{\partial F}{\partial x}\,dx + \frac{\partial F}{\partial y}\,dy = 0$$

This implies that the function F is constant on R and solutions to the

DE are given by the relation

$$F(x,y) = C$$

Recognizing Exactness

There is a theorem from calculus that ensures that if a function F has first partials on a domain, and if those partials are continuous, then the second mixed partials are equal. That is,

$$\frac{\partial^2 F}{\partial y \partial x} = \frac{\partial^2 F}{\partial x \partial y}.$$

If $\dfrac{\partial F}{\partial x} = M$

then

$$\frac{\partial^2 F}{\partial y \partial x} = \frac{\partial M}{\partial y}$$

If it is true that

$$\frac{\partial F}{\partial x} = M \quad \text{and} \quad \frac{\partial F}{\partial y} = N$$

this provides a condition for exactness, namely

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Exact Equations

Theorem: Let M and N be continuous on some rectangle R in the plane. Then the equation

$$M(x,y)\,dx + N(x,y)\,dy = 0$$

is exact if and only if

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$$

Example

Show that the equation is exact and obtain a family of solutions.

$$(2xy - \sec^2 x)\,dx + (x^2 + 2y)\,dy = 0$$

$M(x,y) = 2xy - \sec^2 x$ and $N(x,y) = x^2 + 2y$

$\dfrac{\partial M}{\partial y} = 2x \cdot 1 - 0 = 2x$ $\qquad$ $\dfrac{\partial N}{\partial x} = 2x + 0 = 2x$

Since $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$, the equation is exact. So there exist some function $F(x,y)$ such that solutions are given by the relation.

$$F(x,y) = C$$

We know $\frac{\partial F}{\partial x} = M(x,y) = 2xy - \sec^2 x$

and $\frac{\partial F}{\partial y} = N(x,y) = x^2 + 2y$

$$F(x,y) = \int \frac{\partial F}{\partial x}\,dx = \int (2xy - \sec^2 x)\,dx$$

$$= 2y\frac{x^2}{2} - \tan x + g(y)$$

$$= x^2 y - \tan x + g(y)$$

The "constant" of integration $g(y)$ can depend on y but not x.

We know ① $F(x,y) = x^2 y - \tan x + g(y)$

and ② $\dfrac{\partial F}{\partial y} = x^2 + 2y$

From ① $\dfrac{\partial F}{\partial y} = x^2 \cdot 1 - 0 + g'(y) = x^2 + g'(y)$

Matching gives $g'(y) = 2y$. An anti derivative is $g(y) = y^2$.

$F(x,y) = x^2 y - \tan x + y^2$

the solutions are given implicitly by

$$F(x,y) = C$$

i.e.

$$x^2y - \tan x + y^2 = C \,.$$