# August 24 Math 2306 sec. 57 Fall 2017

#### Section 4: First Order Equations: Bernoulli Equations

Suppose P(x) and f(x) are continuous on some interval (a, b) and n is a real number different from 0 or 1 (not necessarily an integer). An equation of the form

$$\frac{dy}{dx} + P(x)y = f(x)y^n$$

is called a Bernoulli equation.

**Observation:** This equation has the flavor of a linear ODE, but since  $n \neq 0, 1$  it is necessarily nonlinear. So our previous approach involving an integrating factor does not apply directly. Fortunately, we can use a change of variables to obtain a related linear equation.

#### Solving the Bernoulli Equation

$$\frac{dy}{dx} + P(x)y = f(x)y^n \tag{1}$$

Let the new variable 
$$u = y^{1-n}$$
. Find  $\frac{dy}{dx}$ :  
 $\frac{du}{dx} = (1-n)y^{1-n-1}\frac{dy}{dx} = (1-n)y^{n}\frac{dy}{dx}$   
 $\Rightarrow \frac{y^{n}}{1-n}\frac{du}{dx} = \frac{dy}{dx}$  substitute into the ODE  
 $\frac{y^{n}}{1-n}\frac{du}{dx} + P(x)y = f(x)y^{n}$  Divide by  
 $\frac{y^{n}}{1-n}$ 

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$$\frac{du}{dx} + (i-n)P(x) \frac{\partial}{\partial y} = (i-n)f(x)\frac{\partial}{\partial y}$$
Note  $\frac{y}{b} = y^{i-n} = u$ 
The obt in u is
$$\frac{du}{dx} + (i-n)P(x)u = (i-n)f(x)$$
This is fince in u
$$\frac{du}{dx} + P_i(x)u = f_i(x) \text{ where}$$

$$P_i = (i-n)P \quad nd \quad f_i = (i-n)f$$

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Solve this for u. Then since

r = S 1-2 

### Example

Solve the initial value problem  $y' - y = -e^{2x}y^3$ , subject to y(0) = 1.

Here, 
$$n=3$$
, so  $u=\frac{1-3}{3}=\frac{1-2}{3}$ .  

$$\frac{du}{dx}=-2\sqrt{3}\frac{dy}{dx}=3 \quad \frac{1}{2}\sqrt{3}\frac{du}{dx}=\frac{dy}{dx}$$
Substitute
$$-\frac{1}{2}\sqrt{3}\frac{du}{dx}-\gamma=-\frac{2x}{3}\sqrt{3}$$

$$\frac{du}{dx}-(-2)\frac{9}{5^3}=(-2)\left(-\frac{2x}{5}\right)\frac{9}{5^3}$$

$$\frac{1}{5^3}=\sqrt{2}=0$$
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$$\frac{du}{dx} + 2u = 2e^{2x}$$
 solve this

P(x) = 2,  $\mu = e$ f(x) = 2,  $\mu = e$ f(x) = ef(x) = 2,  $\mu = e$ f(x) = ef(x) = ef(x)

$$e^{2x} \frac{du}{dx} + 2e^{x} u = 2e^{2x} e^{2x}$$

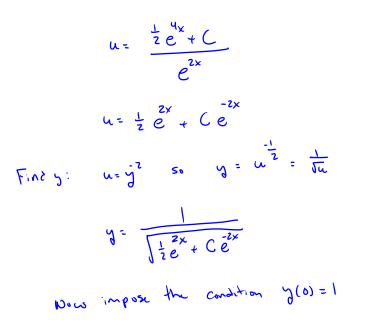
$$\frac{d}{dx} \left[ e^{2x} u \right] = 2 e^{4x}$$

$$\int \frac{d}{dx} \left[ e^{2x} u \right] dx = \int 2e^{4x} dx$$

$$e^{2x}$$
  $u = 2\left(\frac{1}{4}e^{4x}\right) + C$ 

\* Je dx = t ex for dx = t e +( for dr t a t 0

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$$y(0) = \frac{1}{\sqrt{\frac{1}{2}e^{2} + Ce^{2}}} = 1$$

Solve for C 
$$\sqrt{\frac{1}{2} + C} = 1$$
  
 $\frac{1}{2} + C = |^{2} = | = | C = \frac{1}{2}$ 

The solution to the IVP is

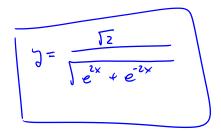
$$y = \frac{1}{\sqrt{\frac{1}{2}e^{2x} + \frac{1}{2}e^{2x}}}$$

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To quoid complex fractions multiply by 12 12

$$y = \frac{1}{\sqrt{\frac{1}{2}e^{2k} + \frac{1}{2}e^{ik}}} \left(\frac{\sqrt{1}}{\sqrt{1}}\right)$$



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#### **Partial Derivatives**

If F(x, y) is a function of two variables, x and y, F may be differentiable with respect to one (or the other or both). A derivative with respect to one variable is called a *partial derivative*, and the partial symbol  $\partial$  is used to denote this. The other variable is held constant. For example,

$$F(x,y)=e^{x^2}\cos(y)$$

has first partial derivatives

$$\frac{\partial F}{\partial x} = 2xe^{x^2}\cos(y)$$
 and  $\frac{\partial F}{\partial y} = -e^{x^2}\sin(y).$ 

If they exist, such a function will have four second partial derivatives

$$\frac{\partial^2 F}{\partial y \partial x}, \quad \frac{\partial^2 F}{\partial x \partial y}, \quad \frac{\partial^2 F}{\partial x^2}, \text{ and } \quad \frac{\partial^2 F}{\partial y^2}.$$

### **Exact Equations**

We considered first order equations of the form

$$M(x, y) dx + N(x, y) dy = 0.$$
 (2)

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The left side is called a *differential form*. We will assume here that M and N are continuous on some (shared) region in the plane.

**Definition:** The equation (2) is called an **exact equation** on some rectangle *R* if there exists a function F(x, y) such that

$$\frac{\partial F}{\partial x} = M(x, y)$$
 and  $\frac{\partial F}{\partial y} = N(x, y)$ 

for every (x, y) in R.

#### **Exact Equation Solution**

If M(x, y) dx + N(x, y) dy = 0 happens to be exact, then it is equivalent to

$$\frac{\partial F}{\partial x}\,dx + \frac{\partial F}{\partial y}\,dy = 0$$

This implies that the function F is constant on R and solutions to the

DE are given by the relation

$$F(x,y)=C$$

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### **Recognizing Exactness**

There is a theorem from calculus that ensures that if a function F has first partials on a domain, and if those partials are continuous, then the second mixed partials are equal. That is,

$$\frac{\partial^2 F}{\partial y \partial x} = \frac{\partial^2 F}{\partial x \partial y}.$$

$$IF \frac{\partial F}{\partial x} = N
 then
 \frac{\partial^2 F}{\partial y \partial x} = \frac{\partial N}{\partial y}$$

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If it is true that

$$\frac{\partial F}{\partial x} = M$$
 and  $\frac{\partial F}{\partial y} = N$ 

this provides a condition for exactness, namely

 $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ 

# **Exact Equations**

**Theorem:** Let M and N be continuous on some rectangle R in the plane. Then the equation

$$M(x,y)\,dx+N(x,y)\,dy=0$$

is exact if and only if

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$$

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#### Example

Show that the equation is exact and obtain a family of solutions.

$$(2xy - \sec^2 x) \, dx + (x^2 + 2y) \, dy = 0$$

$$M(x,y) = 2xy - 5ec^{2}x \quad mc \quad N(x,y) = x^{2}+2y$$

$$\frac{\partial N}{\partial y} = 2x \cdot 1 - 0 = 2x \qquad \frac{\partial N}{\partial x} = 2x + 0 = 2x$$
Since  $\frac{\partial N}{\partial y} = \frac{\partial N}{\partial x}$ , the equation is
exact. So there exist some function  $F(x,y)$ 
such that solutions are given by the relation.
$$F(x,y) = C$$

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We know 
$$\frac{\partial F}{\partial x} = M(x,y) = 2xy - Sec^{2}x$$
  
and  $\frac{\partial F}{\partial y} = N(x,y) = x^{2} + 2y$ 

$$F(x,y) = \int \frac{\partial F}{\partial x} dx = \int (2xy - 5ec^2x) dx$$

$$= 2y \frac{x^{2}}{2} - ton x + g(y)$$
  
=  $x^{2}y - ton x + g(y)$ 

The content of integration g(5) can depend on y but hot X.

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# () F(x,y) = x2y - tax + g(y) We know and $\frac{\partial F}{\partial y} = x^{2} - 0 + g'(y) = x^{2} + g'(y)$ Fron O Matching sines g'(y) = 2y. An anti derivative is $g(y) = y^2$ .

F1377 = x2y - tonx +y2

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is.  $x^2y - tr x + y^2 = C$ .