August 29 Math 3260 sec. 57 Fall 2017

Section 1.7: Linear Independence

We already know that a homogeneous equation $A\mathbf{x} = \mathbf{0}$ can be thought of as an equation in the column vectors of the matrix $A = [\mathbf{a}_1\ \mathbf{a}_2\ \cdots\ \mathbf{a}_n]$ as

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots x_n\mathbf{a}_n = \mathbf{0}.$$

And, we know that at least one solution (the trivial one $x_1 = x_2 = \cdots = x_n = 0$) always exists.

Whether or not there is a nontrivial solution gives us a way to characterize the vectors $\mathbf{a}_1, \ldots, \mathbf{a}_n$.

Definition: Linear Dependence/Independence

An indexed set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_p\}$ in $\mathbb{R}^n$ is said to be **linearly independent** if the vector equation

$$x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \cdots x_p\mathbf{v}_p = \mathbf{0}$$

has only the trivial solution.

The set $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_p\}$ is said to be **linearly dependent** if there exists a set of weights $c_1, c_2, \dots, c_p$ *at least one of which is nonzero* such that

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots c_p\mathbf{v}_p = \mathbf{0}.$$

(i.e. Provided the homogeneous equation possesses a nontrivial solution.)

An equation $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots c_p\mathbf{v}_p = \mathbf{0}$, with at least one $c_i \neq 0$, is called a **linear dependence relation**.

Special Cases

A set with two vectors $\{\mathbf{v}_1, \mathbf{v}_2\}$ is linearly dependent if one is a scalar multiple of the other.

Lin. dependence implies scalars c_1, c_2 such that $c_1\vec{v}_1 + c_2\vec{v}_2 = \vec{0}$ where at least one $c_i \neq 0$.

We can assume $c_1 \neq 0$.

$$c_1\vec{v}_1 + c_2\vec{v}_2 = \vec{0} \implies c_1\vec{v}_1 = -c_2\vec{v}_2$$

$$\implies \vec{v}_1 = \frac{-c_2}{c_1}\vec{v}_2 = k\vec{v}_2$$

where the scalar $k = \frac{-c_2}{c_1}$.

Example

Determine if the set is linearly dependent or linearly independent.

(a) $\mathbf{v}_1 = \begin{bmatrix} 2 \\ 4 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} -1 \\ -2 \end{bmatrix}$ Dependent $\vec{v}_2 = \frac{-1}{2}\vec{v}_1$

we get a linear dependence relation

$\frac{1}{2}\vec{v}_1 + \vec{v}_2 = \vec{0}$

(b) $\mathbf{v}_1 = \begin{bmatrix} 2 \\ 4 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$ Linearly independent

$\vec{v}_1 \neq k\vec{v}_2$ for all

scalers k.

More than Two Vectors

Theorem: The columns of a matrix A are linearly **independent** if and only if the homogeneous equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.

Example: Determine if the set of vectors is linearly dependent or linearly independent. If they are dependent, find a linear dependence relation.

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix},\ \mathbf{v}_2 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix},\ \mathbf{v}_3 = \begin{bmatrix} 4 \\ 1 \\ 0 \end{bmatrix}$$

Form $A = [\vec{v}_1\ \vec{v}_2\ \vec{v}_3]$

$$\begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

This would have a free variable since we have 3 variables and only 2 pivot columns.

To get a linear dependence relationship, do row reduction

$$-2R_2 + R_1 \to R_1$$

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \qquad \begin{aligned} x_1 &= -2x_3 \\ x_2 &= -x_3 \end{aligned}$$

Taking $x_3 = -1$ gives a linear dependence relation

$$2\vec{v}_1 + \vec{v}_2 - \vec{v}_3 = \vec{0}$$

They are linearly dependent.

Theorem

An indexed set of two or more vectors is linearly dependent if and only if at least one vector in the set is a linear combination of the others in the set.

Example: Let **u** and **v** be any nonzero vectors in $\mathbb{R}^3$. Show that if **w** is any vector in Span$\{\mathbf{u}, \mathbf{v}\}$, then the set $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is linearly **dependent.**

Since $\vec{w}$ is in span$\{\vec{u}, \vec{v}\}$,

$\vec{w} = c_1\vec{u} + c_2\vec{v}$ for some

Scalers c_1 and c_2.

Subtract $\vec{w}$ to get

$$c_1\vec{u} + c_2\vec{v} - \vec{w} = \vec{0}$$

This is a linear dependence relation since the coefficient of $\vec{w}$ is $-1 \neq 0$.

Caveat!

A set may be linearly dependent even if all proper subsets are linearly independent. For example, consider

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad \text{and} \quad \mathbf{v}_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

Examine each set $\{\mathbf{v}_1, \mathbf{v}_2\}$, $\{\mathbf{v}_1, \mathbf{v}_3\}$, $\{\mathbf{v}_2, \mathbf{v}_3\}$, and $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$.

The set $\{\vec{v}_1, \vec{v}_2\}$ is lin. independent. $\vec{v}_1 \neq k\vec{v}_2$ for any k or vice versa.

Similarly, $\{\vec{v}_1, \vec{v}_3\}$ is lin. independent as is $\{\vec{v}_2, \vec{v}_3\}$.

$\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is dependent as

$$\vec{v}_3 = \vec{v}_2 - \vec{v}_1 .$$

Two More Theorems

Theorem: If a set contains more vectors than there are entries in each vector, then the set is linearly **dependent**. That is, if $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ is a set of vector in $\mathbb{R}^n$, and $p > n$, then the set is linearly dependent.

e.g. 5 vectors from $\mathbb{R}^3$ or

16 vectors from $\mathbb{R}^{12}$

Theorem: Any set of vectors that contains the zero vector is linearly **dependent**. e.g. $\{\vec{v}_1, \vec{v}_2, \vec{0}\}$ has linear dependence

relation $0\vec{v}_1 + 0\vec{v}_2 + 1\cdot\vec{0} = \vec{0}$

Determine if the set is linearly dependent or linearly independent

(a) $\left\{ \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 3 \\ 3 \\ -5 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 3 \end{bmatrix} \right\}$

4 vectors in $\mathbb{R}^3 \implies$

they are lin. dependent.

Determine if the set is linearly dependent or linearly independent

(b) $\left\{ \begin{bmatrix} 2 \\ 2 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ -8 \\ 1 \end{bmatrix}, \right\}$

This set contains $\vec{0}$ in $\mathbb{R}^4$.

It is lin. dependent.

Section 1.8: Intro to Linear Transformations

Recall that the product $A\mathbf{x}$ is a linear combination of the columns of A—turns out to be a vector. If the columns of A are vectors in $\mathbb{R}^m$, and there are n of them, then

- A is an $m \times n$ matrix,
- the product $A\mathbf{x}$ is defined for $\mathbf{x}$ in $\mathbb{R}^n$, and
- the vector $\mathbf{b} = A\mathbf{x}$ is a vector in $\mathbb{R}^m$.

So we can think of A as an **object that acts** on vectors $\mathbf{x}$ in $\mathbb{R}^n$ (via the product $A\mathbf{x}$) to produce vectors $\mathbf{b}$ in $\mathbb{R}^m$.

Transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$

Definition: A transformation T (a.k.a. **function** or **mapping**) from $\mathbb{R}^n$ to $\mathbb{R}^m$ is a rule that assigns to each vector $\mathbf{x}$ in $\mathbb{R}^n$ a vector $T(\mathbf{x})$ in $\mathbb{R}^m$.

Some relevant terms and notation include

- $\mathbb{R}^n$ is the **domain** and $\mathbb{R}^m$ is called the **codomain.**
- For $\mathbf{x}$ in the domain, $T(\mathbf{x})$ is called the **image** of $\mathbf{x}$ under T.
- The collection of all images is called the **range.**
- The notation $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ may be used to indicate that $\mathbb{R}^n$ is the domain and $\mathbb{R}^m$ is the codomain.
- If $T(\mathbf{x})$ is defined by multiplication by the $m \times n$ matrix A, we may denote this by $\mathbf{x} \mapsto A\mathbf{x}$.

maps to

Matrix Transformation Example

Let $A = \begin{bmatrix} 1 & 3 \\ 2 & 4 \\ 0 & -2 \end{bmatrix}$. Define the transformation $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ by the mapping $T(\mathbf{x}) = A\mathbf{x}$.

(a) Find the image of the vector $\mathbf{u} = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$ under T.

$T(\vec{u}) = A\vec{u} = u_1\vec{a}_1 + u_2\vec{a}_2$ $\qquad A = [\vec{a}_1 \ \vec{a}_2]$

$= 1\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + (-3)\begin{bmatrix} 3 \\ 4 \\ -2 \end{bmatrix}$

$= \begin{bmatrix} 1-9 \\ 2-12 \\ 0+6 \end{bmatrix} = \begin{bmatrix} -8 \\ -10 \\ 6 \end{bmatrix}$

$$A = \begin{bmatrix} 1 & 3 \\ 2 & 4 \\ 0 & -2 \end{bmatrix}$$

(b) Determine a vector **x** in $\mathbb{R}^2$ whose image under T is $\begin{bmatrix} -4 \\ -4 \\ 4 \end{bmatrix}$.

We want to solve $A\vec{x} = \begin{bmatrix} -4 \\ -4 \\ 4 \end{bmatrix}$

Solve the associated system w/ augmented matrix

$$\begin{bmatrix} 1 & 3 & -4 \\ 2 & 4 & -4 \\ 0 & -2 & 4 \end{bmatrix} \qquad -2R_1 + R_2 \to R_2$$

$$\begin{bmatrix} 1 & 3 & -4 \\ 0 & -2 & 4 \\ 0 & -2 & 4 \end{bmatrix}$$

$R_2 + R_3 \to R_3$

then $\frac{-1}{2} R_2 \to R_2$

$$\begin{bmatrix} 1 & 3 & -4 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}$$

$-3R_2 + R_1 \to R_1$

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix} \Longrightarrow \begin{array}{l} x_1 = 2 \\ x_2 = -2 \end{array}$$

a solution is $\vec{x} = \begin{bmatrix} 2 \\ -2 \end{bmatrix}$

$$A = \begin{bmatrix} 1 & 3 \\ 2 & 4 \\ 0 & -2 \end{bmatrix}$$

(c) Determine if $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ is in the range of T. This can be restated as determine if $A\vec{x} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ is solvable, i.e. is the system consistent.

$$\begin{bmatrix} 1 & 3 & 1 \\ 2 & 4 & 0 \\ 0 & -2 & 1 \end{bmatrix} \quad -2R_1 + R_2 \to R_2 \quad \begin{bmatrix} 1 & 3 & 1 \\ 0 & -2 & -2 \\ 0 & -2 & 1 \end{bmatrix}$$

$-R_2 + R_3 \to R_3$

$$\begin{bmatrix} 1 & 3 & 1 \\ 0 & -2 & -2 \\ 0 & 6 & 3 \end{bmatrix}$$

$\uparrow$ pivot column!

The system is inconsistent

hence $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ is not in the range of T.

Linear Transformations

Definition: A transformation T is **linear** provided

(i) $T(\mathbf{u}+\mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ for every $\mathbf{u}.\mathbf{v}$ in the domain of T, and

(ii) $T(c\mathbf{u}) = cT(\mathbf{u})$ for every scalar c and vector $\mathbf{u}$ in the domain of T.

Every matrix transformation (e.g. $\mathbf{x} \mapsto A\mathbf{x}$) is a linear transformation. And it turns out that every linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$ can be expressed in terms of matrix multiplication.

A Theorem About Linear Transformations:

If T is a linear transformation, then

$$T(\mathbf{0}) = \mathbf{0},$$

$$T(c\mathbf{u} + d\mathbf{v}) = cT(\mathbf{u}) + dT(\mathbf{v})$$

for scalars c, d and vectors $\mathbf{u}.\mathbf{v}.$

And in fact

$$T(c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \cdots + c_k\mathbf{u}_k) = T(c_1\mathbf{u}_1) + c_2T(\mathbf{u}_2) + \cdots + c_kT(\mathbf{u}_k).$$

$$= c_1 T(\vec{u}_1) + c_2 T(\vec{u}_2) + \ldots + c_k T(\vec{u}_k).$$

Example

Let r be a nonzero scalar. The transformation $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ defined by

$$T(\mathbf{x}) = r\mathbf{x}$$

is a linear transformation[1].
Show that T is a linear transformation.

We must show
① $T(\vec{u}+\vec{v}) = T(\vec{u}) + T(\vec{v})$
and
② $T(c\vec{u}) = cT(\vec{u})$

For $\vec{u}, \vec{v}$ in $\mathbb{R}^2$

$$T(\vec{u}+\vec{v}) = r(\vec{u}+\vec{v}) = \underbrace{r\vec{u}}_{T(\vec{u})} + \underbrace{r\vec{v}}_{T(\vec{v})} = T(\vec{u}) + T(\vec{v})$$

[1] It's called a **contraction** if $0 < r < 1$ and a **dilation** when $r > 1$

For any scalar c

$$T(c\vec{u}) = r\,c\vec{u} = c\,\underbrace{r\vec{u}}_{T(\vec{u})} = c\,T(\vec{u})$$

So T is a linear transformation.

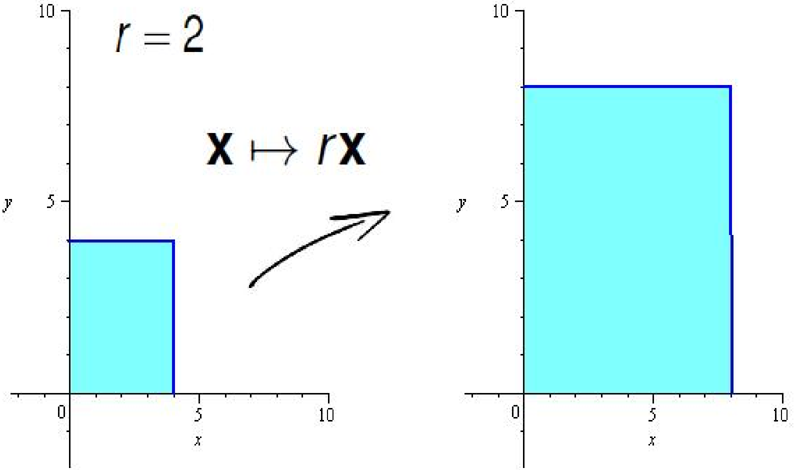


Figure: Geometry of dilation $\mathbf{x} \mapsto 2\mathbf{x}$. The 4 by 4 square maps to an 8 by 8 square.

Section 1.9: The Matrix for a Linear Transformation

Elementary Vectors: We'll use the notation $\mathbf{e}_i$ to denote the vector in $\mathbb{R}^n$ having a 1 in the i^{th} position and zero everywhere else.

e.g. in $\mathbb{R}^2$ the elementary vectors are

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \text{and} \quad \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

in $\mathbb{R}^3$ they would be

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \text{and} \quad \mathbf{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

and so forth.

Note that in $\mathbb{R}^n$, the elementary vectors are the columns of the identity I_n.

Matrix of Linear Transformation

Let $T:\mathbb{R}^2\longrightarrow\mathbb{R}^4$ be a linear transformation, and suppose

$$T(\mathbf{e}_1)=\begin{bmatrix}0\\1\\-2\\4\end{bmatrix},\quad\text{and}\quad T(\mathbf{e}_2)=\begin{bmatrix}1\\1\\-1\\6\end{bmatrix}.$$

Use the fact that T is linear, and the fact that for each $\mathbf{x}$ in $\mathbb{R}^2$ we have

$$\mathbf{x}=\begin{bmatrix}x_1\\x_2\end{bmatrix}=x_1\begin{bmatrix}1\\0\end{bmatrix}+x_2\begin{bmatrix}0\\1\end{bmatrix}=x_1\mathbf{e}_1+x_2\mathbf{e}_2$$

to find a matrix A such that

$$T(\mathbf{x})=A\mathbf{x}\quad\text{for every}\quad\mathbf{x}\in\mathbb{R}^2.$$

$$T(\mathbf{e}_1) = \begin{bmatrix} 0 \\ 1 \\ -2 \\ 4 \end{bmatrix}, \quad \text{and} \quad T(\mathbf{e}_2) = \begin{bmatrix} 1 \\ 1 \\ -1 \\ 6 \end{bmatrix}$$

For any $\vec{x}$ in $\mathbb{R}^2$

$$T(\vec{x}) = T(x_1\vec{e}_1 + x_2\vec{e}_2)$$

$$= x_1 T(\vec{e}_1) + x_2 T(\vec{e}_2)$$

$$= x_1 \begin{bmatrix} 0 \\ 1 \\ -2 \\ 4 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 1 \\ -1 \\ 6 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ -2 & -1 \\ 4 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$T(\vec{e}_1)$ $\quad$ $T(\vec{e}_2)$

So $A = \left[T(\vec{e}_1) \ T(\vec{e}_2) \right]$

Theorem

Let $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be a linear transformation. There exists a unique $m \times n$ matrix A such that

$$T(\mathbf{x}) = A\mathbf{x} \quad \text{for every} \quad \mathbf{x} \in \mathbb{R}^n.$$

Moreover, the j^{th} column of the matrix A is the vector $T(\mathbf{e}_j)$, where $\mathbf{e}_j$ is the j^{th} column of the $n \times n$ identity matrix I_n. That is,

$$A = [T(\mathbf{e}_1) \quad T(\mathbf{e}_2) \quad \cdots \quad T(\mathbf{e}_n)] \, .$$

The matrix A is called the **standard matrix** for the linear transformation T.

Example

Let $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be the scaling trasformation (contraction or dilation for $r > 0$) defined by

$$T(\mathbf{x}) = r\mathbf{x}, \quad \text{for positive scalar } r.$$

Find the standard matrix for T.

$$\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \Rightarrow T(\vec{e}_1) = r \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} r \\ 0 \end{bmatrix}$$

$$\vec{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \Rightarrow T(\vec{e}_2) = r \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ r \end{bmatrix}$$

The standard matrix is

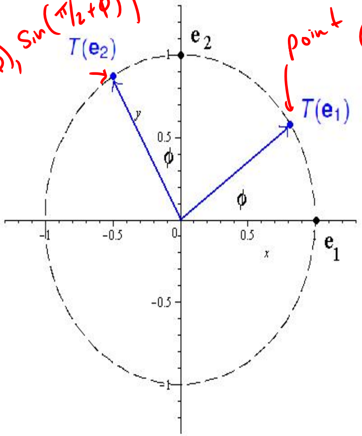
$$A = \begin{bmatrix} r & 0 \\ 0 & r \end{bmatrix}.$$

Example

Let $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be the rotation trasformation that rotates each point in $\mathbb{R}^2$ counter clockwise about the origin through an angle ϕ. Find the standard matrix for T.

point $\left(\cos\left(\frac{\pi}{2}+\phi\right), \sin\left(\pi/2+\phi\right)\right)$

point $(\cos\phi, \sin\phi)$



$$T(\vec{e}_1) = \begin{bmatrix} \cos\phi \\ \sin\phi \end{bmatrix}$$

$$T(\vec{e}_2) = \begin{bmatrix} \cos\left(\frac{\pi}{2}+\phi\right) \\ \sin\left(\frac{\pi}{2}+\phi\right) \end{bmatrix}$$

Recall $\cos\left(\frac{\pi}{2}+\phi\right) = \cos\frac{\pi}{2}\cos\phi - \sin\frac{\pi}{2}\sin\phi$

$$= -\sin\phi$$

$$\sin\left(\frac{\pi}{2}+\phi\right) = \sin\frac{\pi}{2}\cos\phi + \sin\phi\cos\frac{\pi}{2}$$

$$= \cos\phi$$

So the standard matrix

$$A = \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix}$$

Example[2]

Let $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be the projection tranformation that projects each point onto the x_1 axis

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 \\ 0 \end{bmatrix}.$$

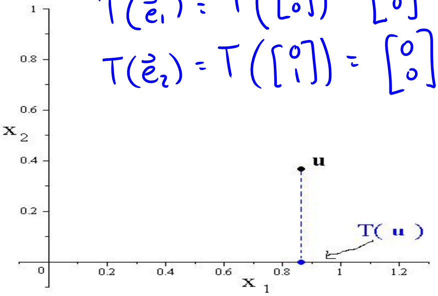
Find the standard matrix for T.

$$T(\vec{e}_1) = T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$T(\vec{e}_2) = T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

So the matrix is

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$



[2] See pages 73–75 in Lay for matrices associated with other geometric tranformation on $\mathbb{R}^2$