Section 1.7: Linear Independence

**Definition** An indexed set of vectors \( \{v_1, v_2, \ldots, v_p\} \) in \( \mathbb{R}^n \) is said to be **linearly independent** if the vector equation

\[
x_1 v_1 + x_2 v_2 + \cdots + x_p v_p = 0
\]

has only the trivial solution.

The set \( \{v_1, v_2, \ldots, v_p\} \) is said to be **linearly dependent** if there exists a set of weights \( c_1, c_2, \ldots, c_p \) at least one of which is nonzero such that

\[
c_1 v_1 + c_2 v_2 + \cdots + c_p v_p = 0.
\]

(i.e. Provided the homogeneous equation possesses a nontrivial solution.)

An equation \( c_1 v_1 + c_2 v_2 + \cdots + c_p v_p = 0 \), with at least one \( c_i \neq 0 \), is called a **linear dependence relation**.
Two or More Vectors

A set with two vectors \( \{ \mathbf{v}_1, \mathbf{v}_2 \} \) is linearly dependent if one is a scalar multiple of the other.

**Theorem:** The columns of a matrix \( A \) are linearly independent if and only if the homogeneous equation \( A \mathbf{x} = \mathbf{0} \) has only the trivial solution.
Example

Determine if the set of vectors is linearly dependent or linearly independent. If they are dependent, find a linear dependence relation.

\[ \mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 4 \\ 1 \\ 0 \end{bmatrix} \]

We can consider the equation \( \mathbf{A} \mathbf{x} = \mathbf{0} \) where \( \mathbf{A} = [\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3] \)

\[
\begin{bmatrix}
1 & 2 & 4 \\
0 & 1 & 1 \\
0 & 0 & 0
\end{bmatrix}
\]

\(2R_2 + R_1 \rightarrow R_1\)

\[
\begin{bmatrix}
1 & 0 & 2 \\
0 & 1 & 1 \\
0 & 0 & 0
\end{bmatrix}
\]

3 variables, 2 pivots, so there is a free variable.
There is a non-trivial solution $\Rightarrow$ the vectors are l.i.n. dependent.

\[ X_1 = -2X_3 \]
\[ X_2 = -X_3 \]
\[ X_3 \text{- free} \]

Taking $X_3 = -1$, then $X_1 = 2$ and $X_2 = 1$

We get a l.i.n. dependence relation

\[ 2\vec{v}_1 + \vec{v}_2 - \vec{v}_3 = \vec{0} \]
Theorem

An indexed set of two or more vectors is linearly dependent if and only if at least one vector in the set is a linear combination of the others in the set.

Example: Let $\mathbf{u}$ and $\mathbf{v}$ be any nonzero vectors in $\mathbb{R}^3$. Show that if $\mathbf{w}$ is any vector in $\text{Span}\{\mathbf{u}, \mathbf{v}\}$, then the set $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is linearly dependent.

\[
\text{Since } \mathbf{w} \text{ is in } \text{Span}\{\mathbf{u}, \mathbf{v}\}, \text{ there exist scalars } c_1, c_2 \text{ such that }
\]
\[
\mathbf{w} = c_1 \mathbf{u} + c_2 \mathbf{v}
\]

Subtract $\mathbf{w}$ to obtain
\[ c_1 \mathbf{u} + c_2 \mathbf{v} - w = 0 \]

The coefficient of \( \mathbf{w} \) is \(-1\neq 0\). Hence this is a linear dependence relation and \( \{\mathbf{u}, \mathbf{v}, \mathbf{w}\} \) is lin. dependent.
Caveat!

A set may be linearly dependent even if all proper subsets are linearly independent. For example, consider

\[
\begin{bmatrix}
1 \\
0 \\
0
\end{bmatrix}, \quad \begin{bmatrix}
1 \\
1 \\
0
\end{bmatrix}, \quad \text{and} \quad \begin{bmatrix}
0 \\
0 \\
1
\end{bmatrix}.
\]

Examine each set \(\{\vec{v}_1, \vec{v}_2\}\), \(\{\vec{v}_1, \vec{v}_3\}\), \(\{\vec{v}_2, \vec{v}_3\}\), and \(\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}\).

\(\{\vec{v}_1, \vec{v}_2\}\) is lin. independent since \(\vec{v}_1 \neq k \vec{v}_2\) for any scalar \(k\) and vice versa.

Similarly \(\{\vec{v}_2, \vec{v}_3\}\) and \(\{\vec{v}_1, \vec{v}_3\}\) are lin. independent.

But \(\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}\) is lin. dependent since

\[
\vec{v}_3 = \vec{v}_2 - \vec{v}_1.
\]
Two More Theorems

**Theorem:** If a set contains more vectors than there are entries in each vector, then the set is linearly dependent. That is, if \( \{v_1, v_2, \ldots, v_p\} \) is a set of vector in \( \mathbb{R}^n \), and \( p > n \), then the set is linearly dependent.

- E.g. 5 vectors in \( \mathbb{R}^2 \)
- or 32 vectors in \( \mathbb{R}^2 \)

**Theorem:** Any set of vectors that contains the zero vector is linearly dependent.
Determine if the set is linearly dependent or linearly independent

(a) \[ \left\{ \begin{bmatrix} 1 & 2 \\ 3 & -5 \\ 0 & -1 \\ 2 & 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 3 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \end{bmatrix} \right\} \]

Lin. dependent. 4 vectors in \( \mathbb{R}^3 \).
Determine if the set is linearly dependent or linearly independent

(b) \[ \left\{ \begin{bmatrix} 2 \\ 2 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ -8 \\ 1 \end{bmatrix} \right\} \]

Lin. Dependent since the \( \mathbf{0} \) is in it.
Section 1.8: Intro to Linear Transformations

Recall that the product $Ax$ is a linear combination of the columns of $A$—turns out to be a vector. If the columns of $A$ are vectors in $\mathbb{R}^m$, and there are $n$ of them, then

- $A$ is an $m \times n$ matrix,
- the product $Ax$ is defined for $x$ in $\mathbb{R}^n$, and
- the vector $b = Ax$ is a vector in $\mathbb{R}^m$.

So we can think of $A$ as an **object that acts** on vectors $x$ in $\mathbb{R}^n$ (via the product $Ax$) to produce vectors $b$ in $\mathbb{R}^m$. 
Transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$

**Definition:** A transformation $T$ (a.k.a. function or mapping) from $\mathbb{R}^n$ to $\mathbb{R}^m$ is a rule that assigns to each vector $\mathbf{x}$ in $\mathbb{R}^n$ a vector $T(\mathbf{x})$ in $\mathbb{R}^m$.

Some relevant terms and notation include

- $\mathbb{R}^n$ is the **domain** and $\mathbb{R}^m$ is called the **codomain**.
- For $\mathbf{x}$ in the domain, $T(\mathbf{x})$ is called the **image** of $\mathbf{x}$ under $T$.
- The collection of all images is called the **range**.
- The notation $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ may be used to indicate that $\mathbb{R}^n$ is the domain and $\mathbb{R}^m$ is the codomain.
- If $T(\mathbf{x})$ is defined by multiplication by the $m \times n$ matrix $A$, we may denote this by $\mathbf{x} \mapsto A\mathbf{x}$. 

$\uparrow$ maps to
Matrix Transformation Example

Let \( A = \begin{bmatrix} 1 & 3 \\ 2 & 4 \\ 0 & -2 \end{bmatrix} \). Define the transformation \( T : \mathbb{R}^2 \rightarrow \mathbb{R}^3 \) by the mapping \( T(\mathbf{x}) = A\mathbf{x} \).

(a) Find the image of the vector \( \mathbf{u} = \begin{bmatrix} 1 \\ -3 \end{bmatrix} \) under \( T \).

\[
T(\mathbf{u}) = A\mathbf{u} = 1 \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + (-3) \begin{bmatrix} 3 \\ 4 \\ -2 \end{bmatrix} = \begin{bmatrix} 1 - 9 \\ 2 - 12 \\ 0 + 6 \end{bmatrix} = \begin{bmatrix} -8 \\ -10 \\ 6 \end{bmatrix}
\]
\[ A = \begin{bmatrix} 1 & 3 \\ 2 & 4 \\ 0 & -2 \end{bmatrix} \]

(b) Determine a vector \( \mathbf{x} \) in \( \mathbb{R}^2 \) whose image under \( T \) is \( \begin{bmatrix} -4 \\ -4 \\ 4 \end{bmatrix} \).

\[ T(\mathbf{x}) = x_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 3 \\ -2 \end{bmatrix} = \begin{bmatrix} -4 \\ -4 \end{bmatrix} \]

if

\[ \begin{array}{c}
\begin{bmatrix}
1 & 3 \\
2 & 4 \\
0 & -2
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2
\end{bmatrix}
= \begin{bmatrix}
-4 \\
-4 \\
4
\end{bmatrix}
\end{array} \]

\begin{align*}
\Rightarrow & \quad x_1 + 3x_2 = -4 \\
& \quad 2x_1 + 4x_2 = -4 \\
& \quad -2x_2 = 4
\end{align*}

solve (if possible)
\[
\begin{bmatrix}
1 & 3 & -4 \\
0 & -2 & 4 \\
0 & -2 & 4
\end{bmatrix} - R_2 + R_3 \rightarrow R_3 \\
\begin{bmatrix}
1 & 3 & -4 \\
0 & -2 & 4 \\
0 & -2 & 4
\end{bmatrix} \text{ then} \\
\frac{1}{2} R_2 \rightarrow R_2 \\
\begin{bmatrix}
1 & 3 & -4 \\
0 & 1 & -2 \\
0 & 0 & 0
\end{bmatrix}
\]

\[ -3R_2 + R_1 \rightarrow R_1 \]

\[
\begin{bmatrix}
1 & 0 & 2 \\
0 & 1 & -2 \\
0 & 0 & 0
\end{bmatrix}
\]

\[ \Rightarrow \]

\[
\begin{align*}
X_1 &= 2 \\
X_2 &= -2
\end{align*}
\]

So \[ \hat{x} = \begin{bmatrix} 2 \\ -2 \end{bmatrix} \]
$$A = \begin{bmatrix} 1 & 3 \\ 2 & 4 \\ 0 & -2 \end{bmatrix}$$

(c) Determine if $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ is in the range of $T$.

Is there $\vec{x}$ such that $T(\vec{x}) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$?

\[
\begin{bmatrix} 1 & 3 & 1 \\ 2 & 4 & 0 \\ 0 & -2 & 1 \end{bmatrix} -2R_1 + R_2 + R_3 \quad \begin{bmatrix} 1 & 3 & 1 \\ 0 & -2 & -2 \\ 0 & -2 & 1 \end{bmatrix}
\]

$-R_2 + R_3 + R_3$
\[
\begin{bmatrix}
1 & 3 & 1 \\
0 & -2 & -2 \\
0 & 0 & 3
\end{bmatrix}
\]

The underlying system is inconsistent.

\[
(1) \text{ is not in the range of } T.
\]
Linear Transformations

**Definition:** A transformation $T$ is **linear** provided

(i) $T(u + v) = T(u) + T(v)$ for every $u, v$ in the domain of $T$, and

(ii) $T(cu) = cT(u)$ for every scalar $c$ and vector $u$ in the domain of $T$.

Every matrix transformation (e.g. $x \mapsto Ax$) is a linear transformation. And it turns out that every linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$ can be expressed in terms of matrix multiplication.
A Theorem About Linear Transformations:

If $T$ is a linear transformation, then

$$T(0) = 0,$$

$$T(cu + dv) = cT(u) + dT(v)$$

for scalars $c, d$ and vectors $u, v$.

And in fact

$$T(c_1 u_1 + c_2 u_2 + \cdots + c_k u_k) = T(c_1 u_1) + c_2 T(u_2) + \cdots + c_k T(u_k).$$

$$= c_1 T(u_1) + c_2 T(u_2) + \cdots + c_k T(u_k).$$
Example

Let $r$ be a nonzero scalar. The transformation $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ defined by

$$T(x) = rx$$

is a linear transformation\(^1\).

Show that $T$ is a linear transformation.

If $\mathbf{u}, \mathbf{v}$ are in $\mathbb{R}^2$

$$T(\mathbf{u} + \mathbf{v}) = r(\mathbf{u} + \mathbf{v}) = r\mathbf{u} + r\mathbf{v} = T(\mathbf{u}) + T(\mathbf{v})$$

\(^1\)It’s called a **contraction** if $0 < r < 1$ and a **dilation** when $r > 1$.
For scalar $c$

$$T(c\hat{u}) = c \hat{u} = c \frac{\hat{u}}{T(\hat{u})} = c T(\hat{u})$$

$T$ satisfies both properties. Hence it is a linear transformation.
Figure: Geometry of dilation $x \mapsto 2x$. The 4 by 4 square maps to an 8 by 8 square.
Section 1.9: The Matrix for a Linear Transformation

**Elementary Vectors:** We’ll use the notation $e_i$ to denote the vector in $\mathbb{R}^n$ having a 1 in the $i^{th}$ position and zero everywhere else.

e.g. in $\mathbb{R}^2$ the elementary vectors are

\[ e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \text{and} \quad e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \]

in $\mathbb{R}^3$ they would be

\[ e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \text{and} \quad e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \]

and so forth.

Note that in $\mathbb{R}^n$, the elementary vectors are the columns of the identity $I_n$. 
Matrix of Linear Transformation

Let \( T : \mathbb{R}^2 \rightarrow \mathbb{R}^4 \) be a linear transformation, and suppose

\[
T(e_1) = \begin{bmatrix}
0 \\
1 \\
-2 \\
4
\end{bmatrix}, \quad \text{and} \quad T(e_2) = \begin{bmatrix}
1 \\
1 \\
-1 \\
6
\end{bmatrix}.
\]

Use the fact that \( T \) is linear, and the fact that for each \( x \) in \( \mathbb{R}^2 \) we have

\[
x = \begin{bmatrix}
x_1 \\
x_2
\end{bmatrix} = x_1 \begin{bmatrix}
1 \\
0
\end{bmatrix} + x_2 \begin{bmatrix}
0 \\
1
\end{bmatrix} = x_2 e_1 + x_2 e_2
\]

to find a matrix \( A \) such that

\[
T(x) = Ax \quad \text{for every} \quad x \in \mathbb{R}^2.
\]
\[ T(e_1) = \begin{bmatrix} 0 \\ 1 \\ -2 \\ 4 \end{bmatrix}, \quad \text{and} \quad T(e_2) = \begin{bmatrix} 1 \\ 1 \\ -1 \\ 6 \end{bmatrix} \]

\[ T(\hat{x}) = T(x_1 \hat{e}_1 + x_2 \hat{e}_2) = x_1 T(\hat{e}_1) + x_2 T(\hat{e}_2) \]

\[ = x_1 \begin{bmatrix} 0 \\ 1 \\ -2 \\ 4 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ -1 \\ -1 \\ 6 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 1 \\ -2 & -1 \\ 4 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = A \hat{x} \]

So the matrix \( A \) should be

\[ A = \begin{bmatrix} T(\hat{e}_1) & T(\hat{e}_2) \end{bmatrix} \]
Theorem

Let \( T : \mathbb{R}^n \rightarrow \mathbb{R}^m \) be a linear transformation. There exists a unique \( m \times n \) matrix \( A \) such that

\[
T(x) = Ax \text{ for every } x \in \mathbb{R}^n.
\]

Moreover, the \( j^{th} \) column of the matrix \( A \) is the vector \( T(e_j) \), where \( e_j \) is the \( j^{th} \) column of the \( n \times n \) identity matrix \( I_n \). That is,

\[
A = \begin{bmatrix} T(e_1) & T(e_2) & \cdots & T(e_n) \end{bmatrix}.
\]

The matrix \( A \) is called the **standard matrix** for the linear transformation \( T \).
Example

Let \( T : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \) be the scaling transformation (contraction or dilation for \( r > 0 \)) defined by

\[
T(x) = rx, \quad \text{for positive scalar } r.
\]

Find the standard matrix for \( T \).

\[
\begin{align*}
\vec{e}_1 &= \begin{bmatrix} 1 \\ 0 \end{bmatrix}, & T(\vec{e}_1) &= r\vec{e}_1 = r \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} r \\ 0 \end{bmatrix} \\
\vec{e}_2 &= \begin{bmatrix} 0 \\ 1 \end{bmatrix}, & T(\vec{e}_2) &= r\vec{e}_2 = r \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ r \end{bmatrix}
\end{align*}
\]

The standard matrix

\[
A = \begin{bmatrix} r & 0 \\ 0 & r \end{bmatrix}.
\]
Example

Let $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be the rotation transformation that rotates each point in $\mathbb{R}^2$ counter-clockwise about the origin through an angle $\phi$. Find the standard matrix for $T$.

\[ T(e_1) = \begin{bmatrix} \cos \phi \\ \sin \phi \end{bmatrix} \]

and

\[ T(e_2) = \begin{bmatrix} \cos (\frac{\pi}{2} + \phi) \\ \sin (\frac{\pi}{2} + \phi) \end{bmatrix} \]
Recall \[ \cos \left( \frac{\pi}{2} + \phi \right) = \cos \frac{\pi}{2} \cos \phi - \sin \frac{\pi}{2} \sin \phi \]
\[= -\sin \phi \]
\[\sin \left( \frac{\pi}{2} + \phi \right) = \sin \frac{\pi}{2} \cos \phi + \sin \phi \cos \frac{\pi}{2} \]
\[= \cos \phi \]

So, \[ T(e_2) = \begin{bmatrix} -\sin \phi \\ \cos \phi \end{bmatrix} \]

The standard matrix \[ A = \begin{bmatrix} \cos \phi & -\sin \phi \\ \sin \phi & \cos \phi \end{bmatrix} \]
Example\(^2\)

Let \( T : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \) be the projection transformation that projects each point onto the \( x_1 \) axis

\[
T \left( \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right) = \begin{bmatrix} x_1 \\ 0 \end{bmatrix}.
\]

Find the standard matrix for \( T \).

The standard matrix

\[
A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.
\]

\( T(\hat{e}_1) = T \left( \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \)

\( T(\hat{e}_2) = T \left( \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \)

\(^2\)See pages 73–75 in Lay for matrices associated with other geometric transformation on \( \mathbb{R}^2 \)
One to One, Onto

**Definition:** A mapping \( T : \mathbb{R}^n \longrightarrow \mathbb{R}^m \) is said to be **onto** \( \mathbb{R}^m \) if each \( b \) in \( \mathbb{R}^m \) is the image of at least one \( x \) in \( \mathbb{R}^n \)—i.e. if the range of \( T \) is all of the codomain.

\[ T \circ \exists \; x \text{ is solvable for all } b \in \mathbb{R}^m \]

**Definition:** A mapping \( T : \mathbb{R}^n \longrightarrow \mathbb{R}^m \) is said to be **one to one** if each \( b \) in \( \mathbb{R}^m \) is the image of at most one \( x \) in \( \mathbb{R}^n \).

\[ T(\bar{x}) = T(\bar{y}) \iff \bar{x} = \bar{y} \]
Determine if the transformation is one to one, onto, neither or both.

\[ T(x) = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \end{bmatrix} x. \]

Check one to one: Suppose \( T(\hat{x}) = T(\hat{y}) \).

Then \( T(\hat{x}) - T(\hat{y}) = \hat{0} \)

\( T(\hat{x} - \hat{y}) = \hat{0} \)

If there is a nontrivial solution, then \( \hat{x} \) doesn't have to equal \( \hat{y} \). The mapping would not be one to one.

Consider \( \begin{bmatrix} 0 & 0 & 2 \\ 0 & 1 & 3 \end{bmatrix} \hat{x} = \hat{0} \).
\[
\begin{bmatrix}
1 & 0 & 2 & 0 \\
0 & 1 & 3 & 0 \\
\end{bmatrix}
\]
3 variables, 2 pivots
so there is a free variable,

\[
\begin{align*}
x_1 &= -2x_3 \\
x_2 &= -3x_3 \\
x_3 &= \text{free} \\
\end{align*}
\]

Verify that

\[
\begin{bmatrix}
2 \\
3 \\
-1 \\
\end{bmatrix}
\text{ and } \begin{bmatrix}
0 \\
0 \\
0 \\
\end{bmatrix}
\]

satisfy \( T(\hat{x}) = T(\hat{y}) \) but

\[
\hat{x} \neq \hat{y}.
\]

The transformation is not one to one.